Minimizing Busy Time in Multiple Machine Real-time Scheduling

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ABSTRACT.
We consider the following fundamental scheduling problem. The input consists of n jobs to be scheduled on a set of machines of bounded capacities. Each job is associated with a release time, a due date, a processing time and demand for machine capacity. The goal is to schedule all of the jobs non-preemptively in their release-time-deadline windows, subject to machine capacity constraints, such that the total busy time of the machines is minimized. Our problem has important applications in power-aware scheduling, optical network design and customer service systems. Scheduling to minimize busy times is APX-hard already in the special case where all jobs have the same (unit) processing times and can be scheduled in a fixed time interval.
Our main result is a 5-approximation algorithm for general instances. We extend this result to obtain an algorithm with the same approximation ratio for the problem of scheduling *moldable* jobs, that requires also to determine, for each job, one of several processing-time vs. demand configurations. Better bounds and exact algorithms are derived for several special cases, including proper interval graphs, intervals forming a clique and laminar families of intervals.

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1 Introduction

Traditional research interest in cluster systems has been high performance, such as high throughput, low response time, or load balancing. In this paper we focus on minimizing machine busy times, a fundamental problem in cluster computing, which aims at reducing power consumption (see, e.g., [21] and the references therein).

Given is a set of n jobs $\mathcal{J} = \{J_1, \ldots, J_n\}$ that need to be scheduled on a set of identical machines, each of which having *capacity* g, for some $g \geq 1$. Each job J has a release time $r(J)$, a due date $d(J)$, a processing time (or, length) $p(J) > 0$ (such that $d(J) \geq r(J) + p(J)$) and a *demand* $1 \leq R(J) \leq g$ for machine capacity; this is the amount of capacity required for processing J on any machine.

A feasible solution schedules each job J on a machine M *non-preemptively* during a time interval $[t(J), t(J) + p(J))$, such that $t(J) \geq r(J)$ and $t(J) + p(J) \leq d(J)$, and the total demand of jobs running at any given time on each machine is at most g. We say that a machine M is *busy* at time t if there is at least one job J scheduled on M such that $t \in [t(J), t(J) + p(J))$; otherwise, M is *idle* at time t. We call the time period in which a machine M is busy its *busy period* and denote its length by $\texttt{busy}(M)$. The goal is to find a feasible schedule of all jobs on a set of machines such that the total busy time of the machines, given by $\sum_M \texttt{busy}(M)$, is minimized. We consider the offline version of this problem where the entire input is given in advance.

Note that the number of machines to be used is part of the output (and can take any integral value $m \geq 1$). Indeed, a solution which minimizes the total busy time may not be optimal in the number of machines used. Also, it is NP-hard to approximate our problem within ratio better than $\frac{3}{2}$, already in the spacial case where all jobs have the same (unit) processing times and can be scheduled in a fixed time interval, by a simple reduction from the subset sum problem.[†]

1.1 Applications

We list below several natural applications of our problem.

Power-aware scheduling The objective of power-aware scheduling is to minimize the power consumption for running a cluster of machines, while supporting Service Level Agreements (SLAs). SLAs, which define the negotiated agreements between service providers and consumers, include quality of service parameters such as demand for a computing resource and a deadline. The power consumption of a machine is assumed to be proportional to the time the machine is in *on* state. While *on*, a machine can process several tasks simultaneously. The number of these tasks hardly affects the power consumption, but must be below the given machine's capacity. Thus, we get an instance of our problem of minimizing the total busy time of the schedule.

[†]Given the integers $a_1, \ldots, a_n \in \{1, \ldots, g\}$ summing to $2g$, the subset sum problem (SSP) is to determine if there is a subset of numbers adding to exactly g. In the reduction, we create for each i, a job J_i with demand a_i, release time 0, processing time 1 and deadline 1. The *yes* instance of SSP results in the busy time of 2 while the *no* instance results in the busy time of 3.

Optical network design Hardware costs in optical networks depend on the usage of switching units such as Optical Add-Drop Multiplexers (or OADMs). Communication is done by *lightpaths*, which are simple paths in the network. A lightpath connecting nodes u and v is using one OADM in each intermediate node. Often, the traffic supported by the network requires transmission rates which are lower than the full wavelength capacity; thus, the network operator has to be able to put together (or, groom) low-capacity demands into the high capacity fibers. Taking g to be the *grooming* factor, for some $g \geq 1$, this can be viewed as grouping the lightpaths with the same wavelength so that at most g of them can share one edge. In terms of OADMs, if g lightpaths of the same wavelength ("color") pass through the same node, they can share a single OADM.

Given a network with a line topology, a grooming factor $g \geq 1$ and a set of lightpaths $\{p_j = (a_j, b_j) | j = 1, ..., n\}$, we need to color the lightpaths such that the number of OADMs is minimized. This yields the following instance of our scheduling problem. There are n unit demand jobs, and machine capacity is g. For $j = 1, \ldots, n$, job J_j needs to be executed in the time interval $[a_j + 1/2, b_j - 1/2]$ (i.e., $p(J_j) = b_j - a_j - 1$, $r(J_j) = a_j + 1/2$ and $d(J_j) = b_j - 1/2$). Grooming up to g lightpaths in the optical network implies that the corresponding jobs are scheduled on the same machine, and vice versa. In other words, different colors in the optical network instance correspond to different machines in the scheduling problem instance, and an OADM at node i corresponds to the interval $[i - 1/2, i + 1/2]$. Thus, the cost of a coloring of the lightpaths is equal to the cost of the schedule of the corresponding set of jobs. This *grooming problem* has become central in optimizing switching costs for optical networks [11, 10, 9].

Unit commitment given future demand The Unit commitment in power systems involves determining the start-up and shut-down schedule of generation units to meet the required demand. This is one of the major problems in power generation (see, e.g., [23, 2] and the references therein). Under-commitment of units would result in extra cost due to the need to purchase the missing power in the spot market, while overcommitment would result in extra operating cost. In a simplified version of the problem, assume that all generation units have the same capacity and that the blocks of future demands are given in advance. This yields an instance of our real-time scheduling problem, where each generation unit corresponds to a machine, and each block of demand corresponds to a job.

1.2 Related Work

Job scheduling on parallel machines has been widely studied (see, e.g., the surveys in [7, 4]). In particular, much attention was given to *interval scheduling* [14], where jobs are given as intervals on the real line, each representing the time interval in which a job should be processed. Each job has to be processed on some machine, and it is commonly assumed that a machine can process a *single* job at any time. Some of the earlier work on interval scheduling considers the problem of scheduling a feasible subset of jobs whose total weight is maximized, i.e., a *maximum weight independent set* (see, e.g., [1] and the survey in [13]).

There is wide literature also on *real-time scheduling*, where each job has to be processed on some machine during a time interval between its release time and due date. Also, there

are studies of real-time scheduling with demands, where each machine has some capacity; however, to the best of our knowledge, all of this prior art refers to objectives other than minimizing the total busy time of the schedule (see, e.g., [1, 17, 5, 6]). There has been earlier work also on the problem of scheduling the jobs on a set of machines so as to minimize the total cost (see, e.g., [3]), but in these works the cost of scheduling each job is *fixed*. In our problem, the cost of scheduling each of the jobs depends on the other jobs scheduled on the same machine in the corresponding time interval; thus, it may change over time and among different machines. Scheduling *moldable* jobs, where each job can have varying processing times, depending on the amount of resources allotted to this job, has been studied using classic measures, such as minimum makespan, or minimum (weighted) sum of completion times (see, e.g., [20, 15] and a comprehensive survey in [19]).

Our study relates also to *batch scheduling* of conflicting jobs, where the conflicts are given as an interval graph. In the *p-batch* scheduling model (see e.g. Chapter 8 in [4]), a set of jobs can be processed jointly. All the jobs in the batch start simultaneously, and the completion time of a batch is the last completion time of any job in the batch. (For known results on batch scheduling, see e.g., [4, 18].) Our scheduling problem differs from batch scheduling in several aspects. While each machine can process (at most) g jobs simultaneously, for some $g \geq 1$, the jobs need not be partitioned to batches, i.e., each job can start at different time. Also, while in known batch scheduling problems the set of machines is given, we assume that *any* number of machines can be used for the solution. Finally, while common measures in batch scheduling refer to the maximum completion time of a batch, or a function of the completion times of the jobs, we consider the total busy times of the machines. Other work on energy minimization consider utilization of machines with variable capacities, corresponding to their voltage consumption [16], and scheduling of jobs with precedence constraints [12, 24].

The complexity of our scheduling problem was studied in [22]. This paper shows that the problem is NP-hard already for $g = 2$, where the jobs are intervals on the line. The paper [9] considers our scheduling problem where jobs are given as intervals on the line with unit demand. For this version of the problem the paper gives a 4-approximation algorithm for general inputs and better bounds for some subclasses of inputs. In particular, the paper presents a 2-approximation algorithm for instances where no interval is properly contained in another (i.e., the input forms a *proper* interval graph), and a $(2 + \varepsilon)$-approximation for bounded lengths instances, i.e., the length (or, processing time) of any job is bounded by some fixed integer d.[‡] A 2-approximation algorithm was given in [9] for instances where any two intervals intersect, i.e., the input forms a clique (see also in [10]). In this paper we improve and extend the results of [9].

1.3 Our Results

Our main result is a 5-approximation algorithm for real-time scheduling of *moldable* jobs. Before summarizing our results, we introduce some notation. Denote by $I(J)$ the interval $[r(J), d(J))$ in which J can be processed.

[‡]A slight modification of the algorithm yields an improved bound of $1 + \varepsilon$, where $\varepsilon > 0$ is an input parameter.

DEFINITION 1. *An instance $\mathcal{J}$ is said to have* interval jobs *if $d(J) = r(J) + p(J)$ holds for all jobs $J \in \mathcal{J}$. An instance $\mathcal{J}$ with interval jobs is called*

- proper *if for any two jobs $J, J' \in \mathcal{J}$, neither $I(J) \subseteq I(J')$ nor $I(J') \subseteq I(J)$ holds.*
- laminar *if the intervals $I(J)$ for all jobs J form a laminar family, i.e., for any two jobs $J, J' \in \mathcal{J}$, we have $I(J) \cap I(J') = \emptyset$ or $I(J) \subset I(J')$, or $I(J') \subset I(J)$.*
- *a* clique *if intervals $I(J)$ for all jobs J form a clique, i.e., for any two jobs $J, J' \in \mathcal{J}$, we have $I(J) \cap I(J') \neq \emptyset$.*

We first prove the following result for instances with interval jobs.

THEOREM 2. *There exists a 5-approximation algorithm for real-time scheduling instances with interval jobs. Furthermore, if the instance is proper, there exists a 2-approximation algorithm.*

We use the above algorithm, as a subroutine, to design our algorithm for the general real-time scheduling problem.

THEOREM 3. *There exists a 5-approximation algorithm for the real-time scheduling problem.*

Next, we consider an extension to real-time scheduling of *moldable* jobs. In this generalization, a job does not have a fixed processing time and demand; rather, it can be scheduled in one of several possible *configurations*. More precisely, for each job $J \in \mathcal{J}$, we have $q \geq 1$ configurations, where configuration i is given by a pair $(p_i(J), R_i(J))$. The problem involves deciding which configuration i_J is to be used in the schedule for each job J. Once a configuration i_J is finalized for a job J, its processing time and demand are given by $p_{i_J}(J)$ and $R_{i_J}(J)$, respectively. We assume that q is polynomially bounded in the input size.

THEOREM 4. *There exists a 5-approximation algorithm for real-time scheduling of* moldable *jobs.*

Finally, we present improved bounds for some cases of instances with interval jobs.

THEOREM 5. *Consider an instance $\mathcal{J}$ consisting of interval jobs with* unit *demands. There exist (i) a polynomial time exact algorithm if $\mathcal{J}$ is laminar, and (ii) a PTAS if $\mathcal{J}$ is a clique.*

The rest of the paper is organized as follows. In §1.4, we start with some preliminary definitions and observations. Theorems 2, 3, 4, and 5 are proved in §2, §3, §4, and §5, respectively. The missing proofs are given in Appendix §A.

1.4 Preliminaries

We will use throughout the paper properties of the interval representation of a given instance.

DEFINITION 6. *Given a time interval $I = [s, t)$, the length of I is $\texttt{len}(I) = t - s$. This extends to a set $\mathcal{I}$ of intervals; namely, the length of $\mathcal{I}$ is $\texttt{len}(\mathcal{I}) = \sum_{I \in \mathcal{I}} \texttt{len}(I)$. We define the span of $\mathcal{I}$ as $\texttt{span}(\mathcal{I}) = \texttt{len}(\cup \mathcal{I})$.*

Note that $\texttt{span}(\mathcal{I}) \leq \texttt{len}(\mathcal{I})$ and equality holds if and only if $\mathcal{I}$ is a set of pairwise disjoint intervals.

Given an instance $\mathcal{J}$ and machine capacity $g \geq 1$, we denote by $\text{OPT}(\mathcal{J})$ the cost of an optimal solution, that is, a feasible schedule in which the total busy time of the machines is minimized. Also, we denote by $\text{OPT}_\infty(\mathcal{J})$ the cost of the optimum solution for the instance $\mathcal{J}$, assuming that the capacity is $g = \infty$. For any job J, let $w(J) = R(J) \cdot p(J)$ denote the total work required by job J, then for a set of jobs $\mathcal{J}$, $w(\mathcal{J}) = \sum_{J \in \mathcal{J}} w(J)$ is the total work required by the jobs in $\mathcal{J}$. The next observation gives two immediate lower bounds for the cost of any solution.

OBSERVATION 7. *For any instance $\mathcal{J}$ and machine capacity $g \geq 1$, the following bounds hold.*

- *The work bound:* $\text{OPT}(\mathcal{J}) \geq \frac{w(\mathcal{J})}{g}$.
- *The span bound:* $\text{OPT}(\mathcal{J}) \geq \text{OPT}_\infty(\mathcal{J})$.

The work bound holds since g is the maximum capacity that can be allocated by a single machine at any time. The span bound holds since the busy-time does not increase by relaxing the capacity constraint.

While analyzing any schedule S that is clear from the context, we number the machines as $M_1, M_2, \ldots$, and denote by $\mathcal{J}_i$ the set of jobs assigned to machine M_i under the schedule S. W.l.o.g., the busy period of a machine M_i is contiguous; otherwise, we can divide the busy period to contiguous intervals and assign the jobs of each contiguous interval to a different machine. Obviously, this will not change the total busy time. Therefore, we say that a machine M_i has a *busy interval* which starts at the minimum start time of any job scheduled on M_i and ends at the maximum completion time of any of these jobs. It follows that the cost of M_i is the length of its busy interval, i.e., $\texttt{busy}(M_i) = \texttt{span}(\mathcal{J}_i)$ for all $i \geq 1$.[§]

2 Interval Scheduling: Thm. 2

2.1 General Instances with Interval Jobs

In this section we present an algorithm for instances with interval jobs, where each job $J \in \mathcal{J}$ may have an arbitrary processing time and any demand $1 \leq R(J) \leq g$. Algorithm *First_Fit_with_Demands* ($FF_\mathcal{D}$), shown in the frame below, divides the jobs into two groups, NARROW and WIDE, as given below. It schedules NARROW and WIDE jobs on distinct sets of machines. The WIDE jobs are scheduled arbitrarily, while NARROW jobs are scheduled greedily by considering them one after the other, from longest to shortest. Each job is scheduled on the first machine it can fit. Let $\alpha \in [0, 1]$ be a parameter to be fixed later.

DEFINITION 8. *For a subset $\mathcal{J}' \subseteq \mathcal{J}$ of jobs, let* $\text{NARROW}(\mathcal{J}') = \{J \in \mathcal{J}' \mid R(J) \leq \alpha \cdot g\}$ *and* $\text{WIDE}(\mathcal{J}') = \{J \in \mathcal{J}' \mid R(J) > \alpha \cdot g\}$.

We show that if $\alpha = 1/4$, the $FF_\mathcal{D}$ is a 5-approximation algorithm. The following is an overview of the analysis. The proof combines the span bound and the work bound given in Observation 7 with the following analysis. For WIDE jobs we use the work bound. Let m denote the total number of machines used for NARROW jobs, let M_i be ith such machine in the order considered by $FF_\mathcal{D}$, and $\mathcal{J}_i$ be the set of NARROW jobs scheduled on M_i. Using

[§]By $\texttt{span}(\mathcal{J}_i)$ we refer to the span of the set of intervals representing the jobs, as scheduled on M_i.

Observation 11, we relate the cost incurred for the jobs in $\mathcal{J}_{i+1}$ to $\texttt{span}(\mathcal{J}_i)$. This relates the overall cost for the jobs in $\mathcal{J} \setminus \mathcal{J}_1$ to $\text{OPT}(\mathcal{J})$, using the work bound. Then we relate the cost incurred for the jobs in $\mathcal{J}_1$ to $\text{OPT}(\mathcal{J})$, using the span bound.

Algorithm ($FF_{\mathcal{D}}$):

(i) Schedule jobs in $\text{WIDE}(\mathcal{J})$ arbitrarily on some machines. Do not use these machines for scheduling any other jobs.

(ii) Sort the jobs in $\text{NARROW}(\mathcal{J})$ in non-increasing order of length, i.e., $p(J_1) \geq \ldots \geq p(J_{n'})$.

(iii) For $j = 1, \ldots, n'$ do:

  (a) Let m denote the number of machines used for jobs in $\cup_{i<j}\{J_i\}$.

  (b) Assign J_j to the first machine that can process it, i.e., find the minimum value of $i : 1 \leq i \leq m$ such that, at any time $t \in I(J_j)$, the total capacity allocated on M_i is at most $g - R(J_j)$.

  (c) If no such machine exists open $(m+1)$th machine and schedule J_j on it.

THEOREM 9. *If $\alpha = 1/4$ then, for any instance $\mathcal{J}$ with interval jobs, we have*

$$FF_{\mathcal{D}}(\mathcal{J}) \leq \text{OPT}_\infty(\mathcal{J}) + 4 \cdot \frac{w(\mathcal{J})}{g} \leq 5 \cdot \text{OPT}(\mathcal{J}).$$

To prove the theorem we bound the costs of the WIDE and NARROW jobs. It is easy to bound the contribution of WIDE jobs to the overall cost. The following lemma follows directly from the definition of WIDE jobs.

LEMMA 10. *The cost incurred by jobs in $\text{WIDE}(\mathcal{J})$ is at most $\sum_{J \in \text{WIDE}(\mathcal{J})} p(J) \leq \frac{w(\text{WIDE}(\mathcal{J}))}{\alpha \cdot g}$.*

The rest of the section is devoted to bounding the cost of the NARROW jobs. The next observation follows from the fact that the *first-fit* algorithm $FF_{\mathcal{D}}$ assigns a job J to machine M_i with $i \geq 2$ only when it could not have been assigned to machines M_k with $k < i$, due to capacity constraints.

OBSERVATION 11. *Let J be a job assigned to machine M_i by $FF_{\mathcal{D}}$, for some $i \geq 2$. For any machine M_k, $(k < i)$, there is at least one time $t_{i,k}(J) \in J$ and a set $s_{i,k}(J)$ of jobs assigned to M_k such that, for every $J' \in s_{i,k}(J)$, (a) $t_{i,k}(J) \in I(J')$, and (b) $p(J') \geq p(J)$. In addition, $R(J) + \sum_{J' \in s_{i,k}(J)} R(J') > g$.*

In the subsequent analysis, we assume that each job $J \in \mathcal{J}_i$, for $i \geq 2$, fixes a unique time $t_{i,i-1}(J)$ and a unique set of jobs $s_{i,i-1}(J) \subseteq \mathcal{J}_{i-1}$. We say that J *blames* jobs in $s_{i,i-1}(J)$.

LEMMA 12. *For any $1 \leq i \leq m-1$, we have $g \cdot \texttt{span}(\mathcal{J}_{i+1}) \leq \frac{3 \cdot w(\mathcal{J}_i)}{1-\alpha}$.*

PROOF. Following Observation 11, for a job $J \in \mathcal{J}_i$, denote by $b(J)$ the set of jobs in $\mathcal{J}_{i+1}$ which blame J, i.e., $b(J) = \{J' \in \mathcal{J}_{i+1} \mid J \in s_{i+1,i}(J')\}$. Let J_L (resp. J_R) be the job with earliest start time (resp. latest completion time) in $b(J)$. Since each job in $b(J)$ intersects J, we have $\texttt{span}(b(J)) \leq p(J) + p(J_L) + p(J_R) \leq 3 \cdot p(J) = 3 \cdot \frac{w(J)}{R(J)}$. Thus,

$$\sum_{J \in \mathcal{J}_i} R(J)\texttt{span}(b(J)) \leq 3 \cdot w(\mathcal{J}_i). \tag{1}$$

Now, we observe that

$$\sum_{J\in\mathcal{J}_i} R(J)\texttt{span}(b(J)) = \int_{t\in\texttt{span}(\mathcal{J}_{i+1})} \sum_{J\in\mathcal{J}_i:t\in\texttt{span}(b(J))} R(J)dt.$$

We bound the right-hand-side as follows. For any $t \in \texttt{span}(\mathcal{J}_{i+1})$, there exists a job $J' \in \mathcal{J}_{i+1}$ with $t \in [r(J'), d(J'))$. For all jobs $J \in s_{i+1,i}(J')$, since $J' \in b(J)$, we have $t \in \texttt{span}(b(J))$. Hence, $\sum_{J\in\mathcal{J}_i:t\in\texttt{span}(b(J))} R(J) \geq \sum_{J\in s_{i+1,i}(J')} R(J)$. By Observation 11 $\sum_{J\in s_{i+1,i}(J')} R(J) > g - R(J') \geq (1-\alpha)\cdot g$. We thus conclude

$$\sum_{J\in\mathcal{J}_i} R(J)\texttt{span}(b(J)) > \texttt{span}(\mathcal{J}_{i+1})\cdot(1-\alpha)\cdot g. \tag{2}$$

From (1) and (2) we get the lemma.

Proof of Theorem 9: The overall cost of the schedule computed by $FF_{\mathcal{D}}$ is the contribution of WIDE jobs and NARROW jobs. Note that the busy time of M_i, for $1 \leq i \leq m$ is exactly $\texttt{busy}(M_i) = \texttt{span}(\mathcal{J}_i)$. Now from Lemmas 10 and 12, we have that the total cost of $FF_{\mathcal{D}}$ is at most

$$\begin{aligned}
\frac{w(\text{WIDE}(\mathcal{J}))}{\alpha\cdot g} + \sum_{i=1}^{m}\texttt{span}(\mathcal{J}_i) &\leq \frac{w(\text{WIDE}(\mathcal{J}))}{\alpha\cdot g} + \texttt{span}(\mathcal{J}_1) + \sum_{i=1}^{m-1}\frac{3\cdot w(\mathcal{J}_i)}{(1-\alpha)\cdot g} \\
&\leq \frac{w(\text{WIDE}(\mathcal{J}))}{\alpha\cdot g} + \text{OPT}_\infty(\mathcal{J}) + \frac{3\cdot w(\text{NARROW}(\mathcal{J}))}{(1-\alpha)\cdot g} \\
&\leq \text{OPT}_\infty(\mathcal{J}) + \max\left\{\frac{1}{\alpha}, \frac{3}{1-\alpha}\right\}\cdot\frac{w(\mathcal{J})}{g} \\
&\leq \text{OPT}_\infty(\mathcal{J}) + 4\cdot\frac{w(\mathcal{J})}{g}.
\end{aligned}$$

The second inequality follows from the span bound, namely, $\texttt{span}(\mathcal{J}_1) \leq \text{OPT}_\infty(\mathcal{J})$. The last inequality holds since $\alpha = 1/4$. The proof now follows from Observation 7. ∎

2.2 Proper Instances with Interval Jobs

In this section we consider instances in which no job interval is contained in another. The intersection graphs for such instances are known as *proper interval graphs*. The simple greedy algorithm consists of two steps. In the first step, the jobs are sorted by their starting times (note that, in a proper interval graph, this is also the order of the jobs by completion times). In the second step the jobs are assigned to machines greedily in a NextFit manner; that is, each job is added to the currently filled machine, unless its addition is invalid, in which case a new machine is opened.

Greedy Algorithm for Proper Interval Graphs

(i) Sort the jobs in non-decreasing order of release times, i.e., $r(J_1) \leq \ldots \leq r(J_n)$.

(ii) For $j = 1, \ldots, n$ do: Assign J_j to the currently filled machine if this satisfies the capacity constraint g; otherwise, assign J_j to a new machine and mark it as being current filled.

THEOREM 13. *Greedy is a 2-approximation algorithm for proper interval graphs.*

PROOF. Let D_t be the total demand of jobs active at time t. Also, let M_t^O denote the number of machines active at time t in an optimal schedule, and let M_t^A be the number of machines active at time t in the schedule output by the algorithm.

CLAIM 14. *For any t, we have $D_t > g \left\lfloor \frac{M_t^A - 1}{2} \right\rfloor$.*

CLAIM 15. *For any t, we have $M_t^O \geq M_t^A/2$.*

Therefore, the cost of the output solution is $\int_{t \in \mathtt{span}(\mathcal{J})} M_t^A dt \leq \int_{t \in \mathtt{span}(\mathcal{J})} 2 \cdot M_t^O dt = 2 \cdot \text{OPT}(\mathcal{J})$, as claimed.

3 Real-time Scheduling: Thm. 3

In this section we show how the results of §2 can extended to scheduling general instances $\mathcal{J}$ where each job J can be processed in the time window $[r(J), d(J))$.

LEMMA 16. *If there exists a β-approximation algorithm for the real-time scheduling with $g = \infty$, there exists an algorithm that computes a feasible solution to the real-time scheduling problem instance, $\mathcal{J}$, with cost at most $\beta \cdot \text{OPT}_\infty(\mathcal{J}) + 4 \cdot \frac{w(\mathcal{J})}{g}$, thus yielding a $(\beta + 4)$-approximation.*

PROOF. We first compute a schedule, called S_∞, with busy-time at most $\beta \cdot \text{OPT}_\infty(\mathcal{J})$, for the given instance with $g = \infty$. Let $[t_\infty(J), t_\infty(J) + p(J)) \subseteq [r(J), d(J))$ be the interval during which job J is scheduled in S_∞. We next create a new instance $\mathcal{J}'$ obtained from $\mathcal{J}$ by replacing $r(J)$ and $d(J)$ with $t_\infty(J)$ and $t_\infty(J) + p(J)$, respectively, for each job J. Note that $\text{OPT}_\infty(\mathcal{J}') \leq \beta \cdot \text{OPT}_\infty(\mathcal{J}) \leq \beta \cdot \text{OPT}(\mathcal{J})$. We then run algorithm $FF_{\mathcal{D}}$ on instance $\mathcal{J}'$. Theorem 9 implies that the resulting solution has busy-time at most $\text{OPT}_\infty(\mathcal{J}') + 4 \cdot \frac{w(\mathcal{J}')}{g} \leq \beta \cdot \text{OPT}_\infty(\mathcal{J}) + 4 \cdot \frac{w(\mathcal{J})}{g} \leq (\beta + 4) \cdot \text{OPT}(\mathcal{J})$ as claimed.

The following theorem with the above lemma implies a 5-approximation algorithm for the real-time scheduling.

THEOREM 17. *If $g = \infty$, the real-time scheduling problem is polynomially solvable.*

The rest of this section is devoted to proving the above theorem. To describe our dynamic programming based algorithm, we first identify some useful properties of the optimum schedule. Recall that we can assume, w.l.o.g., that the busy period of each machine is a contiguous interval.

LEMMA 18. *W.l.o.g., we can assume that the busy period of any machine in the optimum schedule starts at a time given by $d(J) - p(J)$ for some job J and ends at a time given by either $r(J') + p(J')$, for some job J', or $d(J) - p(J) + p(J')$ for some jobs J and J'. Furthermore, we can assume that the start time of any job J is either its release time $r(J)$ or the start time of the busy period of some machine.*

Motivated by Lemma 18, we consider the following definition.

DEFINITION 19. *A time t is called* interesting *if $t = r(J)$ or $d(J) - p(J)$ for some job J, or $t = r(J) + p(J)$ or $d(J) - p(J) + p(J')$ for some jobs J and J'. Let $\mathcal{T}$ denote the set of interesting times.*

Thus, w.l.o.g., we may assume that the busy periods of all machines and placements of all jobs start and end at interesting times. Let the intervals of all the jobs be contained in $[0, T)$. W.l.o.g., we may assume that both 0 and T are interesting times. Note that the number of interesting times is polynomial.

Now we describe our dynamic program. Informally, the algorithm processes the jobs J in the order of non-increasing processing times $p(J)$. It first guesses the placement $[t, t + p(J_1)) \in [r(J_1), d(J_1))$ of job J_1 with largest processing time. Once this is done, the remainder of the problem splits into two *independent* sub-problems: the "left" problem $[0, t)$ and the "right" problem $[t + p(J_1), T)$. This is so because any job J whose interval $[r(J), d(J))$ has an intersection with $[t, t + p(J_1))$ of size at least $p(J)$ can be scheduled inside the interval $[t, t + p(J_1))$ without any extra cost. The "left" sub-problem then estimates the minimum busy time in the interval $[0, t)$ for scheduling jobs whose placement must intersect $[0, t)$; similarly the "right" sub-problem estimates the minimum busy time in the interval $[t + p(J_1), T)$ for scheduling jobs whose placement must intersect $[t + p(J_1), T)$. More formally,

DEFINITION 20. *Let $t_1, t_2 \in \mathcal{T}$ with $t_2 > t_1$ and $\ell = p(J)$ for some job J. Let $\texttt{jobs}[t_1, t_2, \ell]$ denote the set of jobs in $\mathcal{J}$ whose processing time is at most ℓ and whose placement must intersect the interval $[t_1, t_2)$, i.e.,*

$$\texttt{jobs}[t_1, t_2, \ell] = \{J \in \mathcal{J} \mid p(J) \leq \ell, t_1 - r(J) < p(J), d(J) - t_2 < p(J)\}.$$

Let $\texttt{cost}[t_1, t_2, \ell]$ be the minimum busy-time inside the interval $[t_1, t_2)$ for scheduling jobs in $\texttt{jobs}[t_1, t_2, \ell]$ in a feasible manner.

Note that $\texttt{cost}[t_1, t_2, \ell]$ counts the busy-time only inside the interval $[t_1, t_2)$ assuming that the busy-time outside this interval is already "paid for". For convenience, we define $\texttt{jobs}[t_1, t_2, \ell] = \emptyset$ and $\texttt{cost}[t_1, t_2, \ell] = 0$, whenever $t_2 \leq t_1$.

LEMMA 21. *If $\texttt{jobs}[t_1, t_2, \ell] = \emptyset$ then $\texttt{cost}[t_1, t_2, \ell] = 0$. Otherwise, let $J \in \texttt{jobs}[t_1, t_2, \ell]$ be a job with the longest processing time among the jobs in $\texttt{jobs}[t_1, t_2, \ell]$. Then,*

$$\begin{aligned} \texttt{cost}[t_1, t_2, \ell] \quad = \min_{t \in [r(J), d(J) - p(J)) \cap \mathcal{T}} \quad & \Big(\min\{p(J), t + p(J) - t_1, t_2 - t\} \\ & + \texttt{cost}[t_1, t, p(J)] \qquad + \texttt{cost}[t + p(J), t_2, p(J)] \Big). \end{aligned} \tag{3}$$

Note that the number of interesting times and the number of distinct processing lengths are polynomial. Thus, the quantities $\texttt{cost}[t_1, t_2, \ell]$ for $t_1, t_2 \in \mathcal{T}$ and $\ell = p(J)$ for some $J \in \mathcal{J}$ and their corresponding schedules can be computed, using the relation in Lemma 21, in polynomial time. We finally output the schedule corresponding to $\texttt{cost}[0, T, \max_{J \in \mathcal{J}} p(J)]$. By definition, this gives the optimum solution.

4 Real-time Scheduling for Moldable Jobs: Thm. 4

A job J in an instance $\mathcal{J}$ of the real-time scheduling problem with moldable jobs is described by a release time $r(J)$, a due date $d(J)$, and a set of configurations $\{(p_i(J), R_i(J)\}_{i=1,\ldots,q}$. We assume, w.l.o.g., that $p_i(J) \leq d(J) - r(J)$ for all $1 \leq i \leq q$. The goal is to pick a configuration $1 \leq i_J \leq q$ for each job J and schedule these jobs on machines with a capacity g such that the total busy-time is minimized while satisfying the capacity constraints. Given configurations $\vec{i} = \{i_J\}_{J \in \mathcal{J}}$, let $\mathcal{J}(\vec{i})$ denote the instance of real-time scheduling problem derived from $\mathcal{J}$ by fixing configuration i_J for each job J. Let $\text{OPT}(\mathcal{J})$ denote the cost of the optimum solution, and let $\vec{i^*} = \{i_J^*\}_{J \in \mathcal{J}}$ denote the configurations used in the optimum schedule. From Observation 7, we have

$$5 \cdot \text{OPT}(\mathcal{J}) \geq \text{OPT}_\infty(\mathcal{J}(\vec{i^*})) + 4 \cdot \frac{w(\mathcal{J}(\vec{i^*}))}{g}. \tag{4}$$

In this section, we prove the following main lemma.

LEMMA 22. *Given an instance $\mathcal{J}$ of the real-time scheduling with moldable jobs, we can find in polynomial time configurations $\vec{i} = \{i_J\}_{J \in \mathcal{J}}$, such that $\text{OPT}_\infty(\mathcal{J}(\vec{i})) + 4 \cdot \frac{w(\mathcal{J}(\vec{i}))}{g}$ is minimized.*

Now, recall that Lemma 16 and Theorem 17 together imply that given an instance $\mathcal{J}(\vec{i})$ of the real-time scheduling problem, we can compute in polynomial time a feasible schedule with busy-time at most $\text{OPT}_\infty(\mathcal{J}(\vec{i})) + 4 \cdot \frac{w(\mathcal{J}(\vec{i}))}{g}$. Thus, equation (4), Lemma 22, Lemma 16, and Theorem 17 together imply that we can find a schedule with cost at most

$$\text{OPT}_\infty(\mathcal{J}(\vec{i})) + 4 \cdot \frac{w(\mathcal{J}(\vec{i}))}{g} \;\leq\; \text{OPT}_\infty(\mathcal{J}(\vec{i^*})) + 4 \cdot \frac{w(\mathcal{J}(\vec{i^*}))}{g} \;\leq\; 5 \cdot \text{OPT}(\mathcal{J}),$$

thus yielding a 5-approximation. The proof of Lemma 22 is given in Appendix A.

5 Interval Scheduling with Unit Demands: Thm. 5

In this section, we consider instances with interval jobs, where all jobs have unit demands, i.e., $p(J) = d(J) - r(J)$ and $R(J) = 1$.

5.1 Laminar Instances

We show a polynomial time exact algorithm in case the job intervals $I(J)$, for all jobs J, form a laminar family, i.e., for any two jobs $J, J' \in \mathcal{J}$, it holds that $I(J) \cap I(J') = \emptyset$ or $I(J) \subset I(J')$, or $I(J') \subset I(J)$.

Since the job intervals are laminar, the jobs can be represented by a forest F of rooted trees, where each vertex in a tree $T \in F$ corresponds to a job, and a vertex $v(J)$ is an ancestor of a vertex $v(J')$ if and only if $I(J') \subset I(J)$. Let the level of a vertex be defined as follows. Any root of a tree in the forest is at level 1. For all other vertices v, the level of v is 1 plus the level of its parent. Consider an algorithm which assigns jobs in level ℓ to machine $M_{\lceil \ell/g \rceil}$.

THEOREM 23. *The algorithm yields an optimal solution for laminar instances.*

PROOF. Clearly, the algorithm outputs a feasible solution, since at most g jobs are scheduled on any machine at any time. Let M_t (resp., N_t) be the number of active machines (resp., jobs) at time t. Then $M_t = \lceil N_t/g \rceil$. This proves the claim.

5.2 A PTAS and more for Cliques

In the following we show that if all jobs have unit demands, and the corresponding graph is a clique, then the problem can be approximated within factor $1+\varepsilon$, for any $\varepsilon > 0$. Recall that for general instances of job intervals with unit demands the problem is NP-hard already for $g = 2$ [22]. We show that for inputs that form a clique the problem with $g = 2$ is solvable in polynomial time. Finally, we show that the maximum revenue problem is solvable for any $g \geq 1$. In this variant, each job is associated with a profit, the busy intervals of the machines are given, and the goal is to find a feasible schedule of maximum profit.

Since the instance $\mathcal{J}$ forms a clique, there is a time t_0 such that $t_0 \in I(J)$ for all $J \in \mathcal{J}$. The PTAS consists of two main phases. First, it extends the interval lengths, then it finds (using dynamic programming) an optimal schedule of the resulting instance on $m = \lceil n/g \rceil$ machines.

Approximation Scheme for a Clique:

1. Let $c > 1$ be a constant.
2. Let t_0 be such that $t_0 \in J$ for all $J \in \mathcal{J}$. Let $\texttt{left}(J) = t_0 - r(J), \texttt{right}(J) = d(J) - t_0$. Also, let $\texttt{sh}(J) = \min\{\texttt{left}(J), \texttt{right}(J)\}$ and $\texttt{lo}(J) = \max\{\texttt{left}(J), \texttt{right}(J)\}$ be the length of the short (resp. long) segment of J w.r.t. t_0. If $\texttt{sh}(J)/\texttt{lo}(J) \in ((k-1)/c, k/c]$ for some $1 \leq k \leq c$, stretch the short segment to round the ratio to k/c.
3. Partition the jobs into $2c-1$ classes. For $\ell \in \{1, \ldots, c\}$, the ℓ's class consists of all jobs for which $\texttt{sh}(J)/\texttt{lo}(J) = \ell/c$ and $\texttt{left}(J) \geq \texttt{right}(J)$. For $\ell \in \{c+1, \ldots, 2c-1\}$, the ℓ's class consists of all jobs for which $\texttt{sh}(J)/\texttt{lo}(J) = (\ell - c)/c$ and $\texttt{left}(J) < \texttt{right}(J)$. Let n_ℓ be the number of jobs in class ℓ, $1 \leq \ell \leq 2c-1$.
4. For $i \geq 1$, let $C_i(n'_1, \ldots, n'_{2c-1})$ be the minimum cost of scheduling the longest n'_ℓ jobs of class ℓ, for all ℓ, on i machines. Let $m = \lceil n/g \rceil$. Use dynamic programming to find a schedule achieving $C_m(n_1, \ldots, n_{2c-1})$.

In the Appendix we show the next result.

THEOREM 24. *For any $\varepsilon \in (0, 1]$, the scheme with $c = 1/\varepsilon$ is a PTAS for any clique.*

The case $g = 2$: In this case we can solve the problem optimally, using a reduction to minimum-weight perfect matching in a complete graph. Given $\mathcal{J}$, construct a complete graph in which each job corresponds to a vertex, and for every pair i, j, the edge (J_i, J_j) has weight $\texttt{span}(J_i \cup J_j)$. Use Edmond's algorithm [8] to find a minimum-weight perfect matching in the graph.

The Max revenue feasibility problem: In this variant, each job is associated with a profit $w(J)$ that is gained if J is processed. Also, the busy intervals of the machines are given, and

the goal is to find a feasible schedule of a maximum-profit subset of jobs. It is possible to solve the problem for any $g \geq 1$ by reducing it to a min-cost max-flow problem.

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A Some Proofs

Proof of Observation 11: Assume by contradiction that for any $t \in J$ the total capacity allocated to M_k is at most $g - R(J)$. Since the algorithm assigns jobs to machines incrementally (i.e., it never unassigns jobs), this property was true also when $FF_{\mathcal{D}}$ scheduled job J. Thus, J should have been assigned to M_k by $FF_{\mathcal{D}}$, if it was not already assigned to a machine with smaller index. This is a contradiction to the fact that J is assigned to M_i. Thus, there exists a time $t_{i,k}(J)$, such that the set $s_{i,k}(J)$ consisting of the jobs that are assigned to machine M_k when J was scheduled by $FF_{\mathcal{D}}$ that also contain time $t_{i,k}(J)$, satisfies $R(J) + \sum_{J' \in s_{i,k}(J)} R(J') > g$. Property (b) follows since the jobs are considered by the algorithm in a non-increasing order of their length. ∎

Proof of Claim 14: If $M_t^A \leq 2$ then the claim trivially holds. For a given $t > 0$, let $m = M_t^A \geq 3$. The first machine processes at least one job, J, with a positive demand. Since the graph is a proper interval graph, any job J' assigned to each of the next $m-2$ machines starts after J and ends after J, thus, J' is active at time t. Also, all these machines were active at the time the m-th machine is opened, and thus the total demand of jobs on any two consequent machines starting from the second machine is more than g. If $m-2$ is even, $D_t > g\frac{m-2}{2} = g\left\lfloor \frac{m-1}{2} \right\rfloor$. If $m-2$ is odd, the total demand on the $(m-1)$-th machine and the job causing the opening of the m-th machine is more than g and $D_t > g\frac{m-3}{2} + g = g\frac{m-1}{2}$. ∎

Proof of Claim 15: Clearly, for any $t \geq 0$, $M_t^O \geq \lceil D_t/g \rceil$. Using Claim 14, we get that

$$M_t^O \geq \lceil D_t/g \rceil > \left\lfloor \frac{M_t^A - 1}{2} \right\rfloor.$$

Since M_t^O is integral, we get $M_t^O \geq \lfloor (M_t^A - 1)/2 \rfloor + 1 = \lfloor (M_t^A + 1)/2 \rfloor \geq M_t^A/2$. ∎

Proof of Lemma 18: We start with an optimum schedule S and modify it without increasing its busy-time so that it satisfies the given properties. Consider a machine M with busy period starting at time s. Let L denote the set of jobs on M that start at time s. We gradually increase s, if possible, by moving jobs in L to the right without violating any constraints. If this move creates more jobs starting at time s, we add those jobs to L. Note that such a move does not increase the busy-time of machine M. We perform the move till we cannot increase s further. Note that this happens only when some job $J \in L$ cannot be moved to the right due to its due date $d(J)$. Thus, we have $s = d(J) - p(J)$ for this job J. Now let t denote the end time of the busy period of M, and let R denote the set of jobs ending at time t. We now gradually decrease t, if possible, by moving jobs in R to the left without violating any constraints and without decreasing s. As before, if this move creates more jobs ending at time t, we add those jobs to R. Note that such a move does not increase the busy-time of machine M. We perform the move till we cannot decrease t further. Note that this happens only when some job $J' \in R$ cannot be moved to the left either due to its release time $r(J')$ or since it starts at time s. In such cases, either we have $t = r(J') + p(J')$ for this job J' or $t = s + p(J') = d(J) - p(J) + p(J')$ for the jobs J and J'.

Now, consider a job J. Suppose that in the optimum schedule it is scheduled on machine M with start time s. W.l.o.g., we may assume that the start time of job J is $\max\{r(J), s\}$. This completes the proof. ∎

Proof of Lemma 21: We first show that $\texttt{cost}[t_1, t_2, \ell]$ is at most the expression (3), by constructing a feasible schedule of jobs in $\texttt{jobs}[t_1, t_2, \ell]$ with total busy-time in $[t_1, t_2)$ equal to (3). Consider the job J and time t that achieves the minimum in (3). Schedule J in $[t, t + p(J))$. This job contributes $\min\{p(J), t + p(J) - t_1, t_2 - t\}$ to the busy-time inside $[t_1, t_2)$, depending on whether both $t, t + p(J) \in [t_1, t_2)$, $t \notin [t_1, t_2)$, or $t + p(J) \notin [t_1, t_2)$. Since $p(J) \geq p(J')$ for any other job $J' \in \texttt{jobs}[t_1, t_2, \ell]$, we get that $\texttt{jobs}[t_1, t, p(J)] \cap \texttt{jobs}[t + p(J), t_2, p(J)] = \emptyset$. We use the solution of cost $\texttt{cost}[t_1, t, p(J)]$ (resp., $\texttt{cost}[t + p(J), t_2, p(J)]$) to schedule jobs in $\texttt{jobs}[t_1, t, p(J)]$ (resp., $\texttt{jobs}[t + p(J), t_2, p(J)]$) in the interval $[t_1, t)$ (resp., $[t + p(J), t_2)$). The jobs in $\texttt{jobs}[t_1, t_2, \ell] \setminus \{\{J\} \cup \texttt{jobs}[t_1, t, p(J)] \cup \texttt{jobs}[t + p(J), t_2, p(J)]\}$ can be scheduled inside the interval $[t, t + p(J))$ without paying extra cost. Thus the overall cost of the solution is the expression (3). From the definition of $\texttt{cost}[t_1, t_2, \ell]$, we get that $\texttt{cost}[t_1, t_2, \ell]$ is at most the expression (3), as claimed.

Now we show that $\texttt{cost}[t_1, t_2, \ell]$ is at least the expression (3). Consider the schedule S of cost $\texttt{cost}[t_1, t_2, \ell]$, and let $J \in \texttt{jobs}[t_1, t_2, \ell]$ be a job with the largest processing time. Let t be the start time of J in S. Then S pays the cost $\min\{p(J), t + p(J) - t_1, t_2 - t\}$ inside $[t_1, t_2)$, depending on whether both $t, t + p(J) \in [t_1, t_2)$, $t \notin [t_1, t_2)$, or $t + p(J) \notin [t_1, t_2)$. W.l.o.g., we may assume that S schedules the jobs in $\texttt{jobs}[t_1, t_2, \ell] \setminus (\{J\} \cup \texttt{jobs}[t_1, t, p(J)] \cup \texttt{jobs}[t + p(J), t_2, p(J)])$ inside the interval $[t, t + p(J))$. The cost of S for scheduling jobs $\texttt{jobs}[t_1, t, p(J)]$ (respectively, $\texttt{jobs}[t + p(J), t_2, p(J)]$) in the interval $[t_1, t)$ (respectively, $[t + p(J), t_2)$) is, by definition, at least $\texttt{cost}[t_1, t, p(J)]$ (respectively, $\texttt{cost}[t + p(J), t_2, p(J)]$). Thus $\texttt{cost}[t_1, t_2, \ell]$ is at least the expression (3), as claimed.

This completes the proof. ∎

Proof of Lemma 22: Motivated by Lemma 18 and Definition 19, we define the set of interesting times as follows.

DEFINITION 25. *A time t is called* interesting *if $t = r(J)$ or $d(J) - p_i(J)$ for some job J and configuration i, or $t = r(J) + p_i(J)$ or $d(J) - p_i(J) + p_{i'}(J')$ for some jobs J and J' and their respective configurations i and i'. Let $\mathcal{T}$ denote the set of interesting times.*

Note that the size of $\mathcal{T}$ is polynomial and we can assume, w.l.o.g., that the busy periods of all machines and placements of all jobs start and end at interesting times. Let the intervals of all the jobs be contained in $[0, T)$. W.l.o.g. we can assume that both 0 and T are interesting times. For a job $J \in \mathcal{J}$ and a configuration $1 \leq i_J \leq q$, let $w_{i_J}(J) = p_{i_J}(J) \cdot R_{i_J}(J)$.

DEFINITION 26. *Let $t_1, t_2 \in \mathcal{T}$ with $t_2 > t_1$ and $\ell = p_i(J)$ for some job J. Let*

$$\texttt{jobs}[t_1, t_2, \ell] = \{J \in \mathcal{J} \mid r(J) > t_1 - \ell, d(J) < t_2 + \ell\}.$$

For a choice of configurations $\vec{i} = \{i_J\}_{J \in \texttt{jobs}[t_1, t_2, \ell]}$, let $\texttt{cost}[t_1, t_2, \ell, \vec{i}]$ denote the minimum busy-time inside interval $[t_1, t_2)$ for scheduling jobs in $\texttt{jobs}[t_1, t_2, \ell](\vec{i})$ in a feasible manner. Let $\texttt{ub}[t_1, t_2, \ell]$ be the minimum value of

$$\texttt{cost}[t_1, t_2, \ell, \vec{i}] + 4 \cdot \frac{w(\texttt{jobs}[t_1, t_2, \ell](\vec{i}))}{g},$$

where the minimum is taken over all configurations $\vec{i}$ that satisfy $p_{i_J}(J) \leq \ell$, for all jobs $J \in \texttt{jobs}[t_1, t_2, \ell]$.

As before, $\texttt{cost}[t_1, t_2, \ell, \vec{i}]$ counts the busy-time only inside the interval $[t_1, t_2)$, assuming that the busy-time outside this interval is already "paid for".

LEMMA 27. *If* $\texttt{jobs}[t_1, t_2, \ell] = \emptyset$ *we have* $\texttt{ub}[t_1, t_2, \ell] = 0$. *If* $t_2 \leq t_1$ *then*

$$\texttt{ub}[t_1, t_2, \ell] = \sum_{J \in \texttt{jobs}[t_1, t_2, \ell]} \min_{i_J : p_{i_J}(J) \leq \ell} 4 \cdot \frac{w_{i_J}(J)}{g}.$$

Otherwise, we have

$$\begin{aligned}
\texttt{ub}[t_1, t_2, \ell] \quad = \quad & \min_{J \in \texttt{jobs}[t_1, t_2, \ell]} \; \min_{i_J : p_{i_J}(J) \leq \ell} \; \min_{t \in [r(J), d(J) - p_{i_J}(J)) \cap \mathcal{T}} \\
& \Bigg(\min \left\{ p_{i_J}(J), \max\{0, t + p_{i_J}(J) - t_1\}, \max\{0, t_2 - t\} \right\} \\
& + \sum_{\substack{J' \in \texttt{jobs}[t_1, t_2, \ell] \setminus \\ (\texttt{jobs}[t_1, \min\{t, t_2\}, p_{i_J}(J)] \cup \texttt{jobs}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)])}} \; \min_{i_{J'} : p_{i_{J'}}(J') \leq p_{i_J}(J)} 4 \cdot \frac{w_{i_{J'}}(J')}{g} \\
& + \texttt{ub}[t_1, \min\{t, t_2\}, p_{i_J}(J)] \\
& + \texttt{ub}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)] \Bigg). \qquad (5)
\end{aligned}$$

Note that the number of interesting times and the number of distinct processing lengths are polynomial. Thus, the quantities $\texttt{ub}[t_1, t_2, \ell]$ for $t_1, t_2 \in \mathcal{T}$ and $\ell = p_i(J)$, for some $J \in \mathcal{J}, 1 \leq i \leq q$ and their corresponding job-configurations and schedules can be computed, using the relation in Lemma 27, in polynomial time. The algorithm finally outputs the job-configurations corresponding to $\texttt{ub}[0, T, \max_{J \in \mathcal{J}, 1 \leq i \leq q} p_i(J)]$. By definition, this proves Lemma 22.

Proof of Lemma 27: By definition, it follows that if $\texttt{jobs}[t_1, t_2, \ell] = \emptyset$, we have $\texttt{ub}[t_1, t_2, \ell] = 0$. If $t_2 \leq t_1$, the jobs in $\texttt{jobs}[t_1, t_2, \ell]$ can be scheduled in the interval $[t_1 - \ell, t_2 + \ell)$ without paying any additional busy-time. Therefore, $\texttt{cost}[t_1, t_2, \ell] = 0$, and $\texttt{ub}[t_1, t_2, \ell] = \sum_{J \in \texttt{jobs}[t_1, t_2, \ell]} \min_{i_J : p_{i_J}(J) \leq \ell} 4 \cdot \frac{w_{i_J}(J)}{g}$ follows from the definition.

The rest of the proof is similar to that of Lemma 21.

We first show that $\texttt{ub}[t_1, t_2, \ell]$ is at most the expression (5), by constructing a feasible schedule of jobs in $\texttt{jobs}[t_1, t_2, \ell]$ with total busy-time in $[t_1, t_2)$ equal to (5). Consider the job J, its configuration i_J, and time t that achieve the minimum in (5). Schedule J, fixing its configuration i_J, in $[t, t + p_{i_J}(J))$. This job contributes $\min\left\{p_{i_J}(J), \max\{0, t + p_{i_J}(J) - t_1\}, \max\{0, t_2 - t\}\right\}$ to the busy-time inside $[t_1, t_2)$ depending on whether both $t, t + p_{i_J}(J) \in [t_1, t_2)$, $t \notin [t_1, t_2)$ and $t + p_{i_J}(J) \in [t_1, t_2)$, $t \in [t_1, t_2)$ and $t + p_{i_J}(J) \notin [t_1, t_2)$, or both $t, t + p_{i_J}(J) \notin [t_1, t_2)$. All jobs $J' \in \texttt{jobs}[t_1, t_2, \ell] \setminus (\texttt{jobs}[t_1, \min\{t, t_2\}, p_{i_J}(J)] \cup \texttt{jobs}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)])$ can either be scheduled inside $[t, t + p_{i_J}(J))$ or outside $[t_1, t_2)$, provided we chose configuration $i_{J'}$ satisfying $p_{i_{J'}}(J') \leq p_{i_J}(J)$ for each such job J'. Thus, these jobs do not contribute anything additional to the busy-time inside $[t_1, t_2)$. However

such jobs J' contribute $\min_{i_{J'}:p_{i_{J'}}(J')\leq p_{i_J}(J)} 4 \cdot \frac{w_{i_{J'}}(J')}{g}$ to $\texttt{ub}[t_1, t_2, \ell]$. We next use the solution of cost $\texttt{ub}[t_1, \min\{t, t_2\}, p_{i_J}(J)]$ (respectively, $\texttt{ub}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)]$) to schedule jobs in $\texttt{jobs}[t_1, \min\{t, t_2\}, p_{i_J}(J)]$ (respectively, $\texttt{jobs}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)]$) in the interval $[t_1, \min\{t, t_2\})$ (respectively, $[\max\{t + p_{i_J}(J), t_1\}, t_2)$). Thus the overall cost of the solution is the expression (5). From the definition of $\texttt{ub}[t_1, t_2, \ell]$, we get that $\texttt{ub}[t_1, t_2, \ell]$ is at most the expression (5), as claimed.

Now, we show that $\texttt{ub}[t_1, t_2, \ell]$ is at least the expression (5). Consider the schedule S and configurations $\{i_J\}_{J\in\texttt{jobs}[t_1,t_2,\ell]}$ corresponding to $\texttt{ub}[t_1, t_2, \ell]$, and let $J \in \texttt{jobs}[t_1, t_2, \ell]$ and its configuration i_J be the job with the largest processing time $p_{i_J}(J)$. Let t be the start time of J in S. Thus, the cost upper bound pays $\min\left\{p_{i_J}(J), \max\{0, t + p_{i_J}(J) - t_1\}, \max\{0, t_2 - t\}\right\}$ inside $[t_1, t_2)$. W.l.o.g., we can assume that S schedules the jobs in

$$\texttt{jobs}[t_1, t_2, \ell] \setminus (\texttt{jobs}[t_1, \min\{t, t_2\}, p_{i_J}(J)] \cup \texttt{jobs}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)])$$

inside the interval $[t, t + p_{i_J}(J))$, or outside the interval $[t_1, t_2)$. These jobs J', w.l.o.g., contribute $\min_{i_{J'}:p_{i_{J'}}(J')\leq p_{i_J}(J)} 4 \cdot \frac{w_{i_{J'}}(J')}{g}$ to the cost upper bound. The cost upper bound for scheduling jobs $\texttt{jobs}[t_1, \min\{t, t_2\}, p_{i_J}(J)]$ (resp., $\texttt{jobs}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)]$) in the interval $[t_1, \min\{t, t_2\})$ (resp., $[\max\{t + p_{i_J}(J), t_1\}, t_2)$) is, by definition, at least $\texttt{ub}[t_1, \min\{t, t_2\}, p_{i_J}(J)]$ (resp., $\texttt{ub}[\max\{t + p_{i_J}(J), t_1\}, t_2, p_{i_J}(J)]$). Thus, $\texttt{ub}[t_1, t_2, \ell]$ is at least the expression (5), as claimed.

This completes the proof. ∎

B Analysis of the PTAS for a Clique

We prove Theorem 24 using the following results.

OBSERVATION 28. *Given a clique, for any $c \geq 1$ and $1 \leq \ell \leq 2c - 1$, extending the short segment of any interval in class ℓ may increase the busy time of the schedule at most by factor $1 + 1/c$.*

As stated in step 4, the dynamic programming only considers schedules in which the jobs from each class are assigned to machines from longest to shortest. That is, when scheduling $\hat{n}_\ell$ jobs from class ℓ on machine i, the DP selects the $\hat{n}_\ell$ longest unscheduled jobs in this class. Note that since the jobs of a class share the same $sh(J)/lo(J)$ ratio and the same long side, then the intervals from each class, when sorted according to length, are nested in each other. The following Lemma assures that there exists an optimal schedule of the extended intervals in which the jobs are assigned to machines according to the nesting order.

LEMMA 29. *(Nesting property) Given a clique of intervals, there exists an optimal schedule in which the machines can be ordered such that for any class $1 \leq \ell \leq 2c - 1$ and $1 \leq i < m$, every job of class ℓ assigned to machine i is longer than every job of class ℓ assigned to machine $i + 1$.*

PROOF. We show that if the nesting property does not hold in some schedule, then it is possible to achieve it without increasing the total cost of the schedule. Note that the nesting property is a combination of two properties:

$P1$: The jobs of class ℓ assigned to one machine form a contiguous subset in the nested order of class ℓ.

$P2$: For all $1 \leq i < m$, the set of jobs of class ℓ assigned to machine i precedes the set of jobs of class ℓ assigned to machine $i+1$ in the nested order of class ℓ.

Consider a given schedule. First, we show how to convert the schedule into a one in which property 1 holds: assume that for some ℓ, machine i is assigned the k-th and the $(k+\delta)$-th job from class ℓ (for some $\delta > 1$) but not the $(k+1)$-st job. It is possible to exchange the assignments of the $(k+1)$-st and the $(k+\delta)$-th jobs without increasing the schedule cost (recall that the $(k+\delta)$-th job is included in the $(k+1)$-st, which is included in the k-th job). By performing such inner-class exchanges for each class according to the nesting order, we get a schedule in which a contiguous subset of jobs from each class is scheduled on each machine.

Next we show that it is possible to convert a schedule fulfilling $P1$ into a one fulfilling $P2$. We say that the pair of machines M' and M'' have *inversion* if there exist two classes, j and k, such that M' contains a set of short jobs from j, A'_j, and a set of long jobs from k, A'_k, and M'' contains a set of long jobs from j, A''_j, and a set of short jobs from k, A''_k. By "short" and "long" we refer to the containment relation that we defined on the jobs in each class, i.e., all the jobs in A'_j are contained in the shortest job in A''_j (and similarly for A''_k and A'_k).

Let $h = \min\{|A'_j|, |A''_k|\}$, namely, h is the smaller between the sets of short jobs. W.l.o.g., suppose that the smaller is A'_j. We move the jobs in A'_j to M'' and move the h longest jobs in A''_k to M'. This does not affect the total busy time, since we add to each machine a set of jobs that are contained in some job that is scheduled on this machine. In the resulting schedule, all of the above jobs of class j are scheduled on M', and we have decreased the number of inversions. If, as a result of the new assignment, the jobs of classes j, k assigned to M' or M'', do not form a contiguous subset in the nested order of their class, i.e., $P1$ is violated, then apply inner-class exchanges to close the gap and to keep the relative order of jobs from each class across the machines. These shifts guarantee that $P1$ holds and also, that no new inversions are created when one inversion is removed.

We continue decreasing the number of inversions for pairs of machines and for pairs of classes, until no inversions exist. At this point, it is possible to order the machines in a way that fulfills the nesting property. ∎

The dynamic programming proceeds as follows: For $1 \leq \ell \leq 2c-1$, for any $n' \leq n_\ell$, and $m \leq n_\ell - n'$, let $J_\ell(n')$ be the n' longest job in class ℓ, and let $J_\ell(n', m) = J_\ell(n'+m) \setminus J_\ell(n')$ be the set of m jobs at rank $n'+1, \ldots, n'+m$ in class ℓ. In particular, if $m = 0$ then $J_\ell(n', m) = \emptyset$. For any $(n'_1, \ldots, n'_{2c-1})$ such that $n'_\ell \leq n_\ell$, and any $(m_1, \ldots, m_{2c-1})$ such that $\sum_\ell m_\ell = g$ and $0 \leq m_\ell \leq n_\ell - n'_\ell$, let $M(n'_1, \ldots, n'_{2c-1}; m_1, \ldots, m_{2c-1}) = \texttt{span}(J_1(n'_1, m_1) \cup \cdots \cup J_\ell(n'_{2c-1}, m_{2c-1}))$. Note that $M(n'_1, \ldots, n'_{2c-1}; m_1, \ldots, m_\ell)$ is the busy time of the machine that is assigned m_ℓ jobs from class ℓ starting after the n'_ℓ longest job. Obviously, for $(m_1, \ldots, m_{2c-1})$ such that $\sum_\ell m_\ell = g$, $C_1(m_1, \ldots, m_{2c-1}) = M(0, \ldots, 0; m_1, \ldots, m_{2c-1})$. For $i > 1$, let

$$C_i(n'_1, \ldots, n'_{2c-1}) = \min_{(m_1,\ldots,m_{2c-1}) s.t \sum_\ell m_\ell = g} \{C_{i-1}(n'_1 - m_1, \ldots, n'_{2c-1} - m_{2c-1}) + M(n'_1 - m_1, \ldots, n'_{2c-1} - m_{2c-1}; m_1, \ldots, m_\ell)\}.$$

Since c is a constant, the number of entries in the DP table is polynomial in n, g and the time required to calculate each entry is $n^{O(c)}$. The optimal schedule of the extended instance for m machines is given by $C_m(n_1, \ldots, n_{2c-1})$. The DP schedules exactly g jobs on each machine. If n is not a multiple of g then it is possible to add dummy jobs of length $\varepsilon \to 0$ into the c-th class (around t_0).

The proof of Theorem 24 follows from combining Observation 28, with the above DP. The following claim completes the proof.

CLAIM 30. *If the instance is a clique then any optimal schedule uses exactly $m = \lceil n/g \rceil$ machines.*

PROOF. Clearly, at least m machines are active at time t_0. Assume that $m' > \lceil n/g \rceil$ machines are used in some schedule. We show that the number of machines can be reduced without increasing the total busy time. We say that a machine is *full* if it processes g jobs. Assume that there are k non-full machines, $M_1, M_2, ..., M_k$. Let the busy intervals of these non-full machines be $[s_1, e_1], [s_2, e_2], \ldots, [s_k, e_k]$ such that $s_1 \leq \ldots \leq s_k$. W.l.o.g., these intervals form a proper interval graph, that is, $e_1 \leq \ldots \leq e_k$, else, if for some i, j, $[s_i, e_i]$ is contained in $[s_j, e_j]$ then it is possible to move jobs from M_i to M_j, until j is full or i is empty.

Since $m' > \lceil n/g \rceil$, it holds that $m' \geq (n+g)/g$ and $n \leq g(m'-1)$. The full machines hold exactly $(m'-k)g$ jobs. Thus, at most $n - (m'-k)g \leq g(k-1)$ jobs are scheduled on the k non-full machines. Given that $s_1 \leq \ldots \leq s_k$ and $e_1 \leq \ldots \leq e_k$ and that all k machines together hold at most $(k-1)g$ jobs, We show how to reassign the jobs scheduled on the non full machines, on at most $k-1$ machines without increasing the total busy time:

First, move the rightmost jobs (those with the largest $d(J)$) from M_2 to M_1, until M_1 holds g jobs or M_2 is empty. The busy time of M_1 is increased by $e_2 - e_1$. If M_2 becomes empty we are done (the busy time of M_2 is 0 and we have one less active machine as needed). Next, if M_2 still has jobs on it, move jobs from M_3 to M_2. By the same argument, the busy time of M_2 is increased by $e_3 - e_2$. Continue 'filling machines' until some machine becomes empty. Since the total number of jobs on non-full machines is at most $(k-1)g$, this must happen in M_k at the latest. Let M_j be the machine that becomes empty. The total increase in the busy time of all involved machines is at most $(e_2 - e_1) + (e_3 - e_2) + (e_4 - e_3) + \cdots + (e_j - e_{j-1}) = e_j - e_1$. On the other hand, machine M_j is not active anymore, so its busy time of $e_j - s_j > e_j - e_1$ is saved. It holds that $e_1 > s_j$ since the instance is a clique. We conclude that the total busy time saved is larger than the total added busy time. ∎