Multiple-Source Multiple-Sink Maximum Flow in Directed Planar Graphs in Near-Linear Time

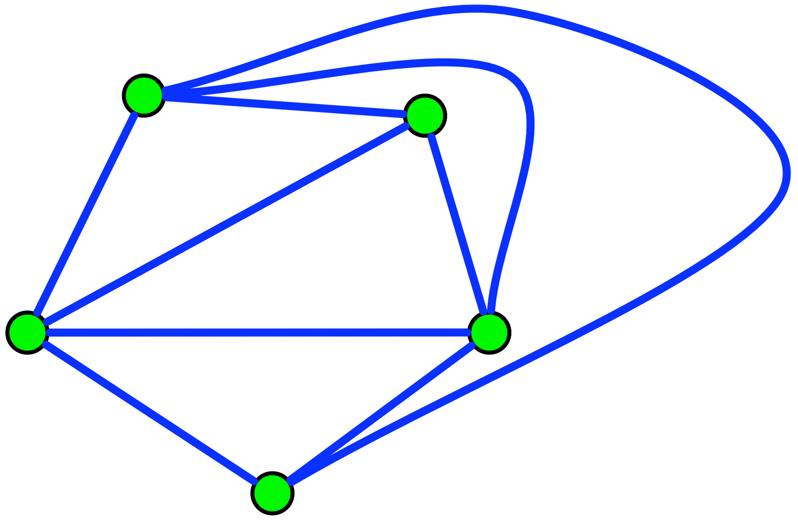
Shay Mozes

BROWN

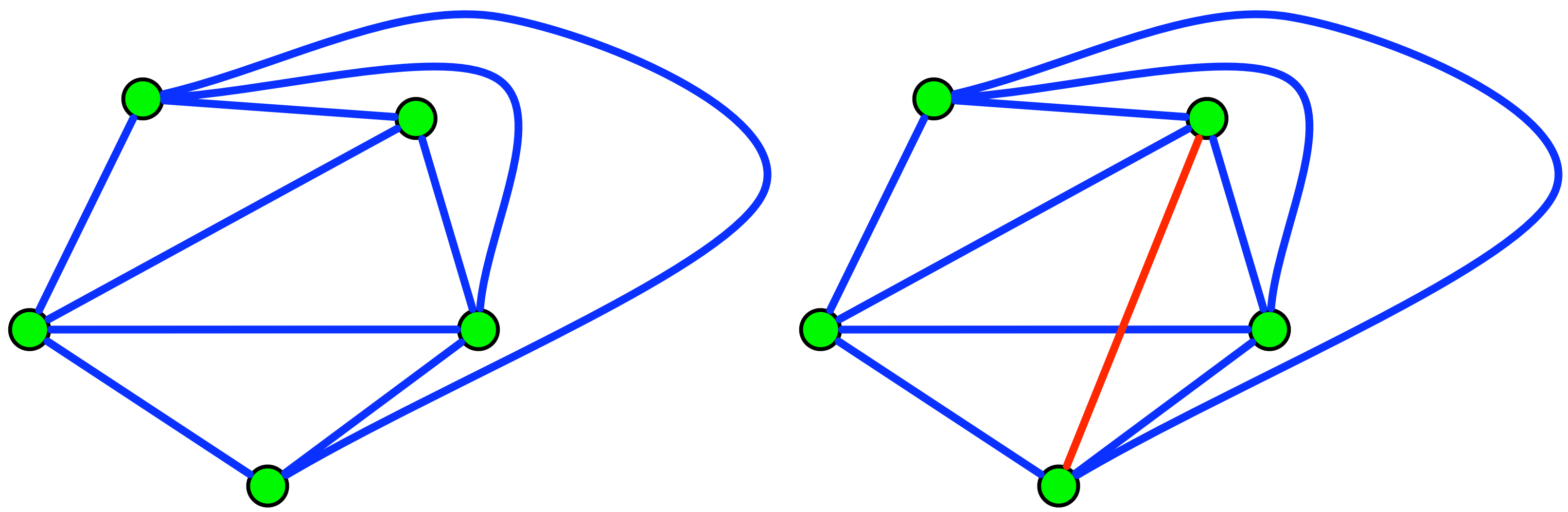
joint work with Cora Borradaile, Philip Klein, Yahav Nussbaum and Christian Wulff-Nilsen

Planar Graphs

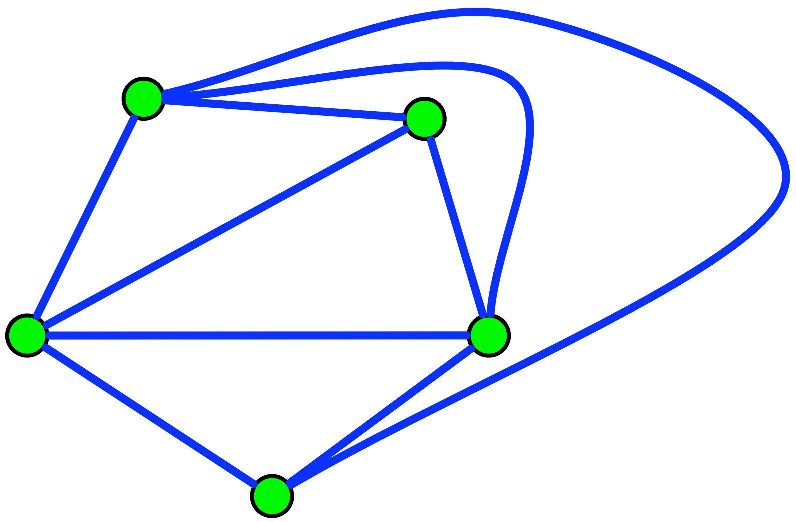
Planar Graphs

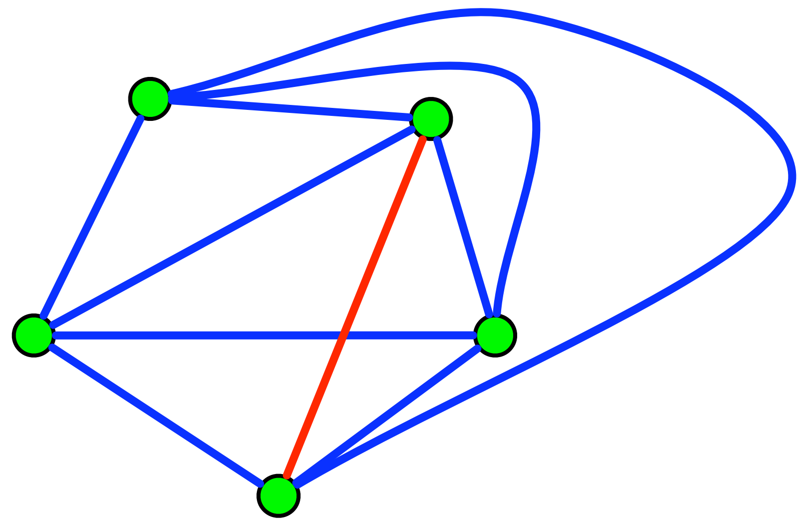


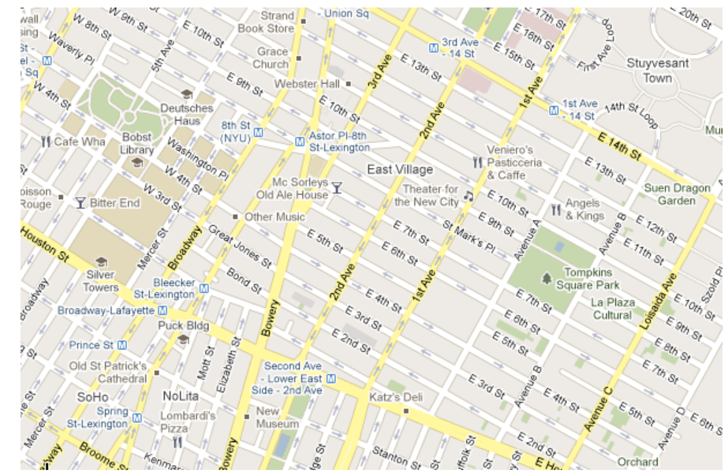
Planar Graphs



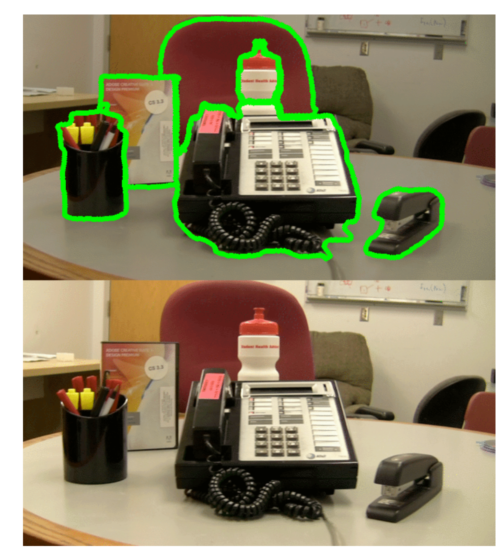
Planar Graphs

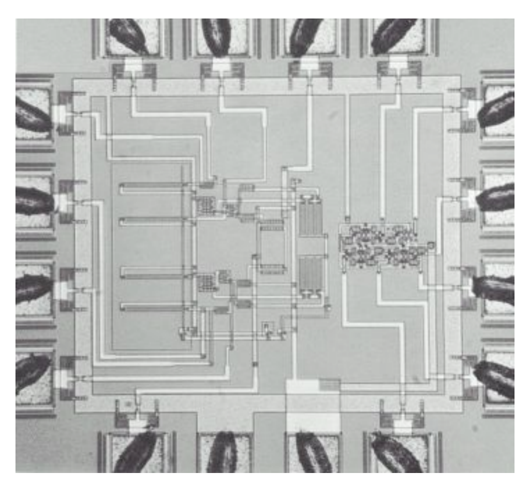




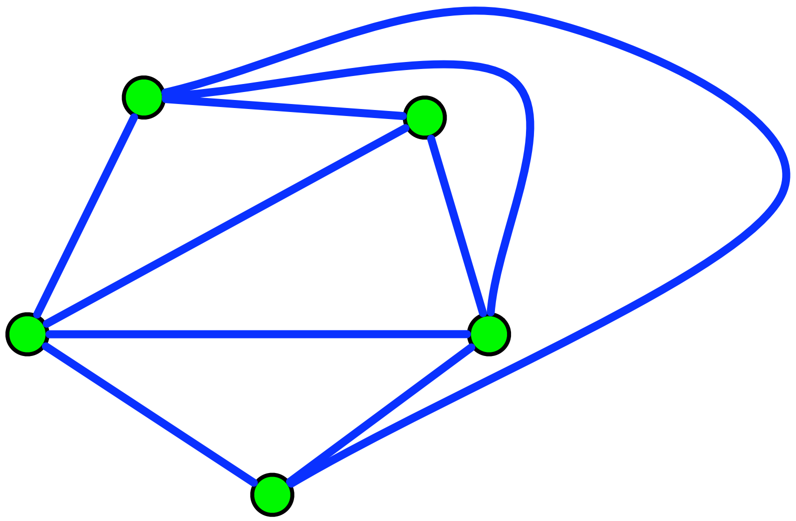


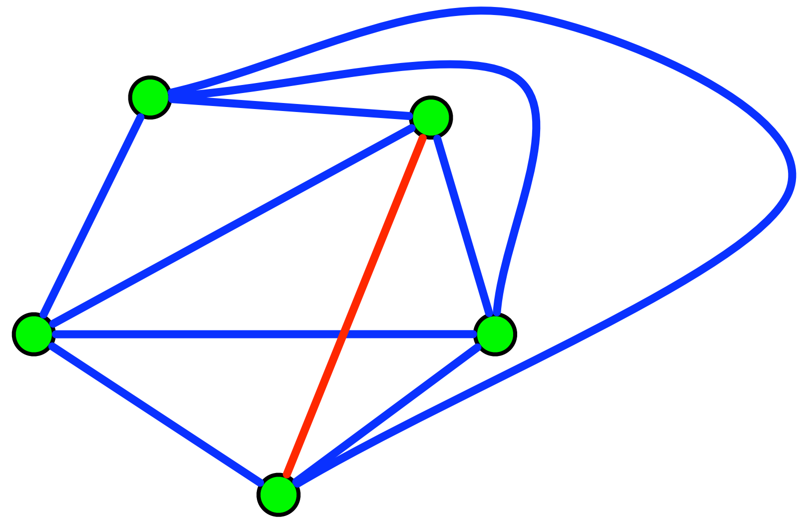
- arise in many applications

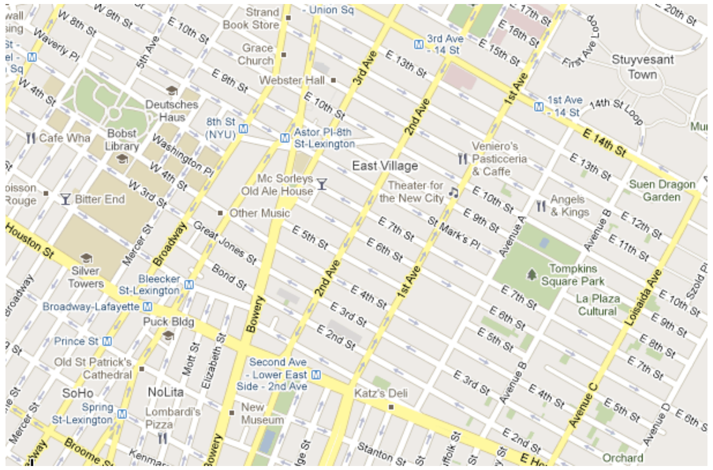




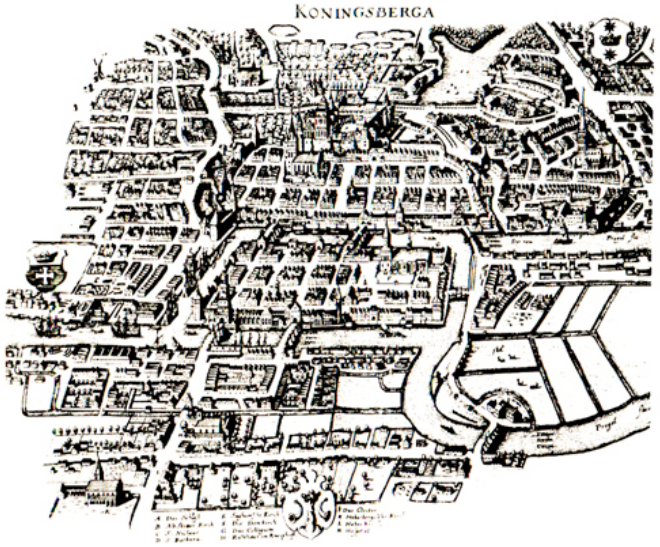
Planar Graphs

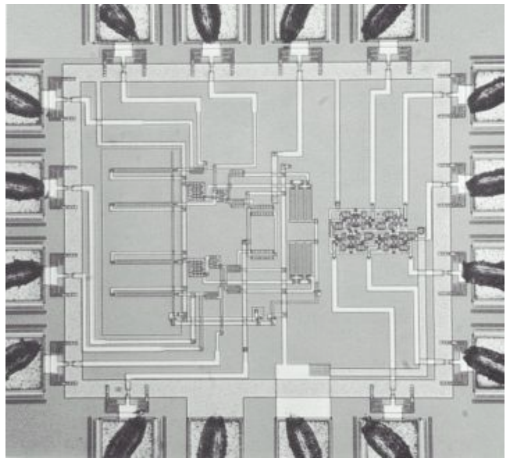




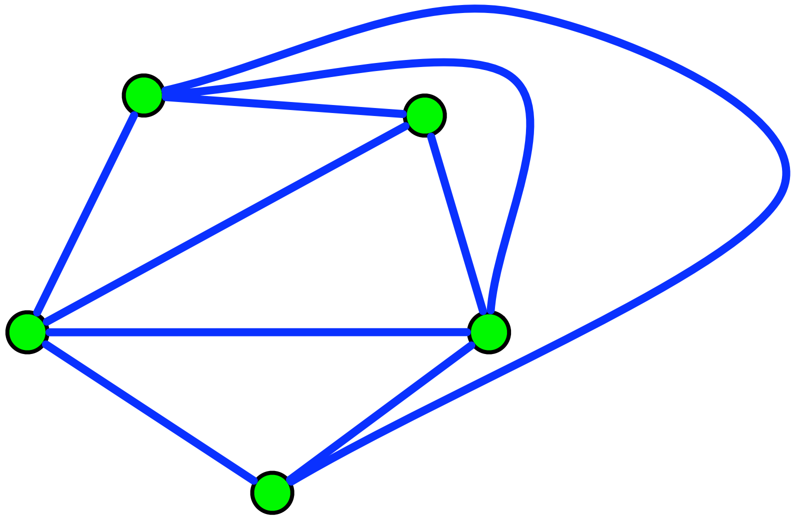


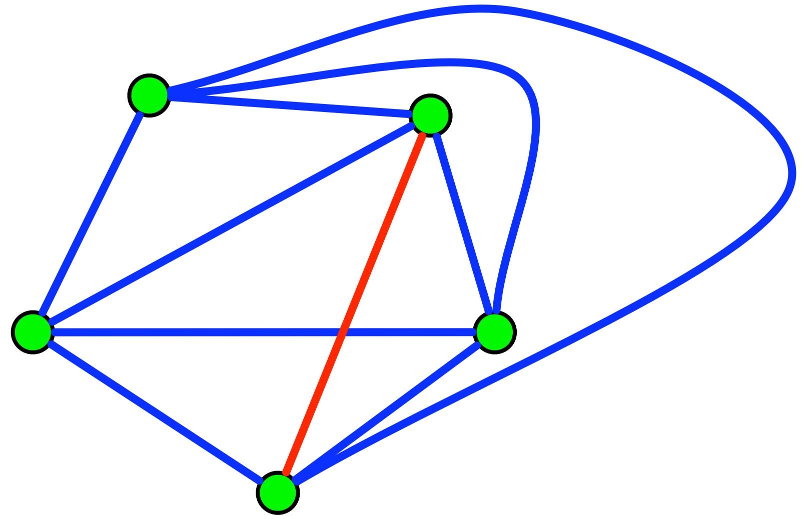
- arise in many applications
- admit faster algorithms

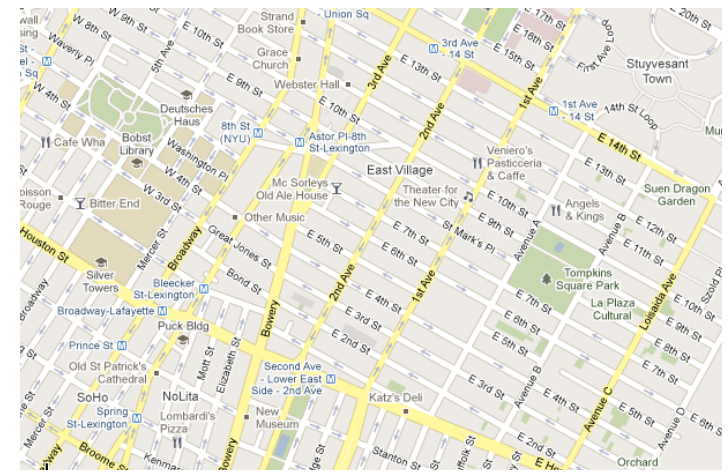




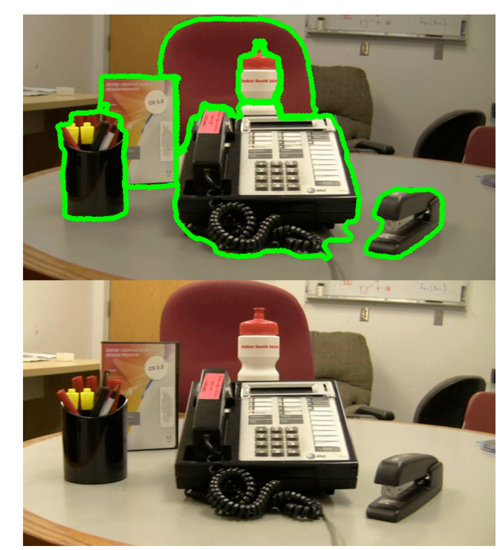
Planar Graphs

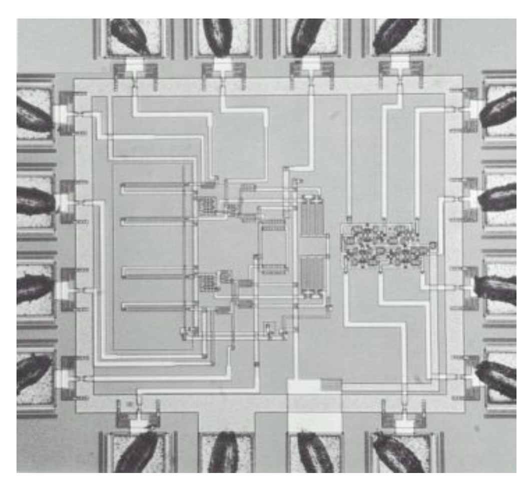




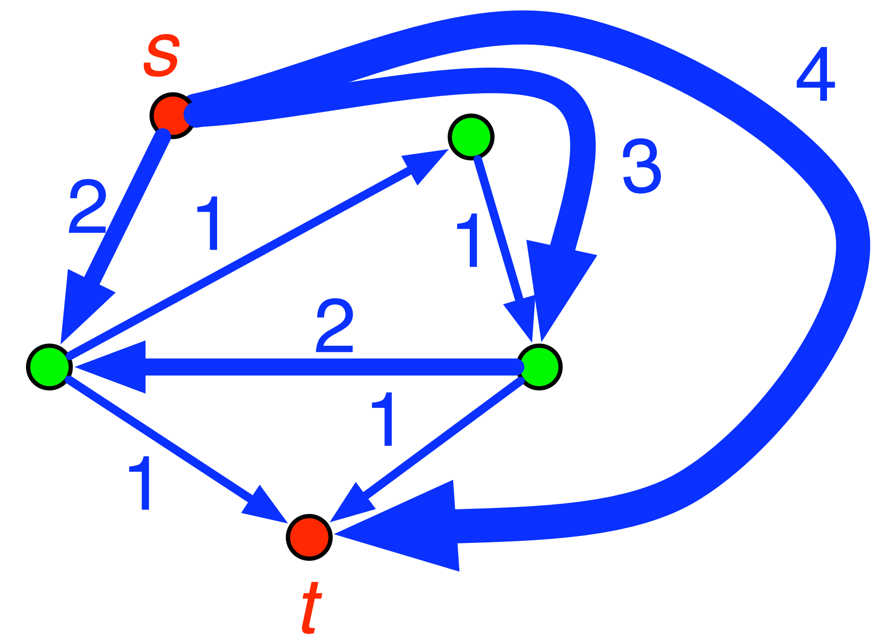


- arise in many applications
- admit faster algorithms
- interesting structural properties





Maximum Flow

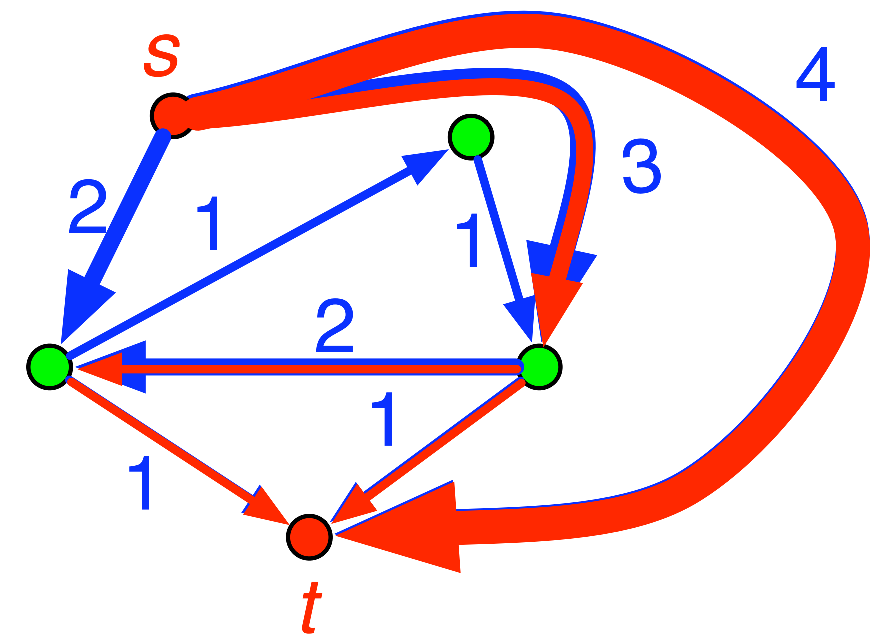


input: a graph G with arc capacities and nodes s,t

output: an assignment of flow to arcs such that:

- conservation at non-terminals
- respects capacity at all arcs
- maximizes the amount of flow entering t

Maximum Flow



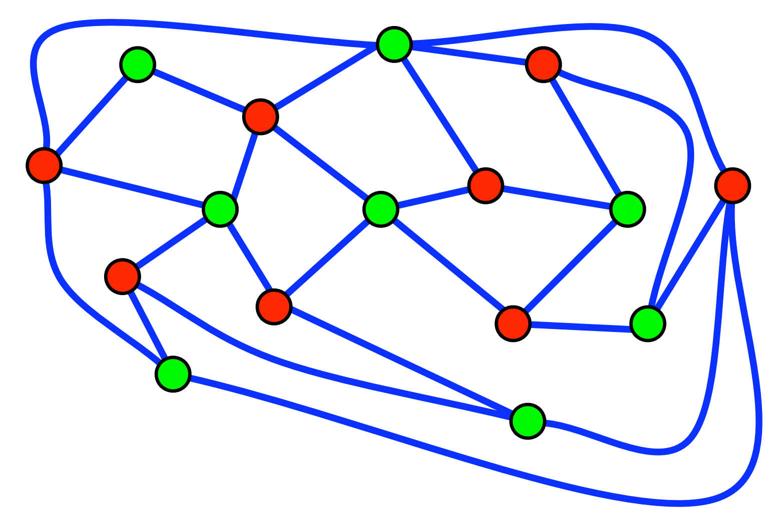
input: a graph G with arc capacities and nodes s,t

output: an assignment of flow to arcs such that:

- conservation at non-terminals
- respects capacity at all arcs
- maximizes the amount of flow entering t

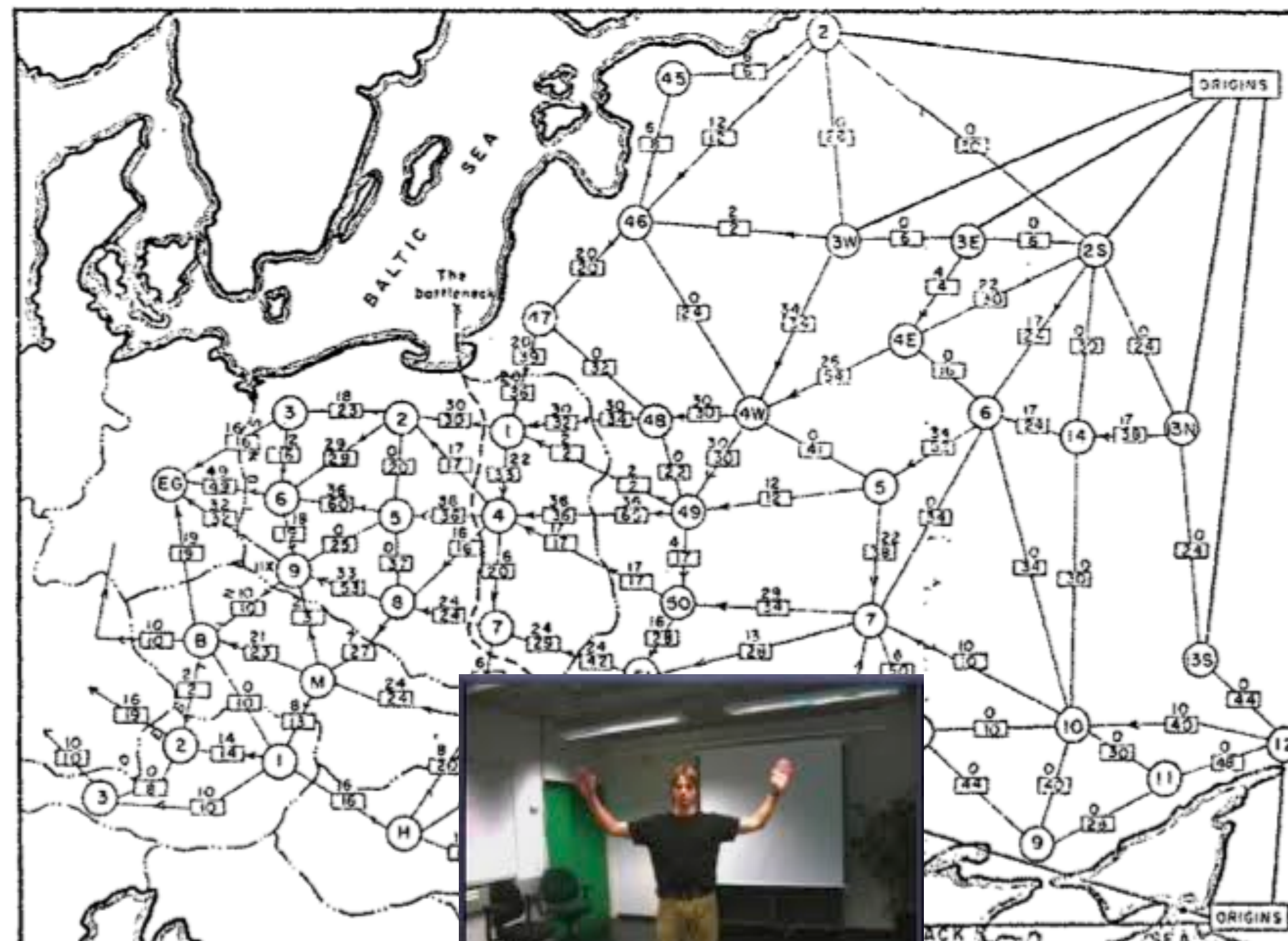
Main Result

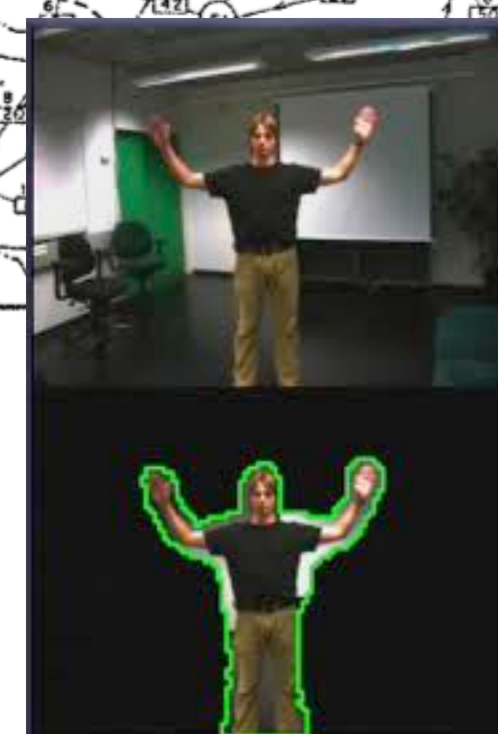
multiple-source, multiple-sink maximum flow in directed planar graphs in $O(n \log^3 n)$ time.



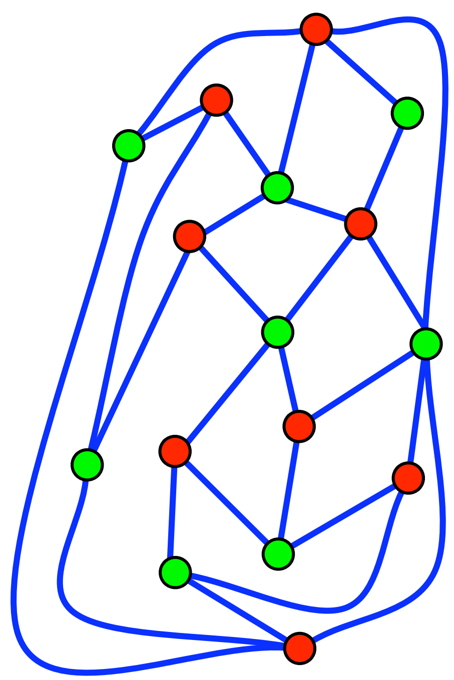
Applications
Multiple Sources and Sinks

- transportation networks (Soviet railroad system)
- computer vision - image segmentation, restoration, stereo, object recognition, texture synthesis (grid)
- maximum bipartite matching

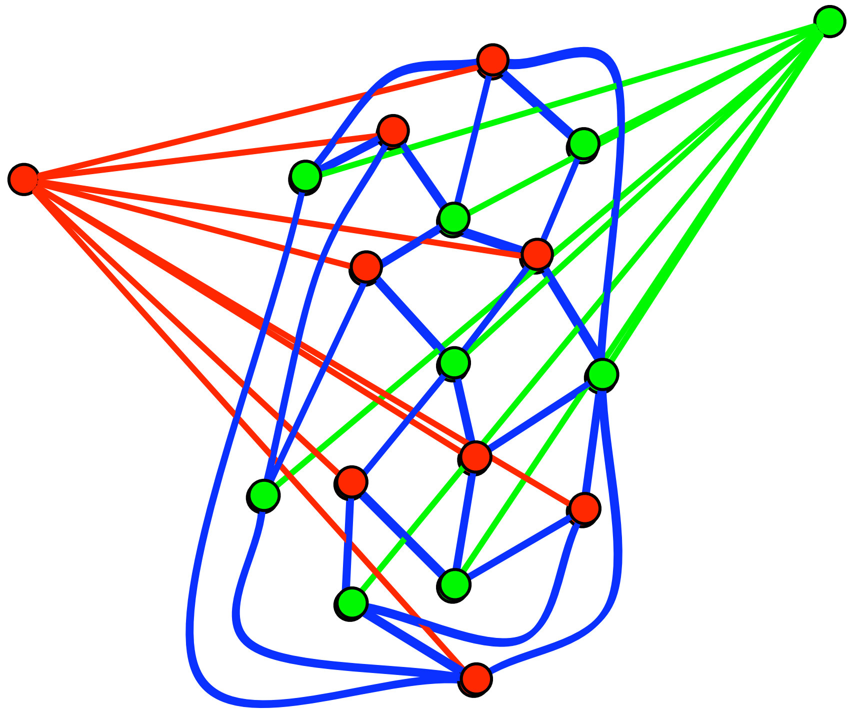




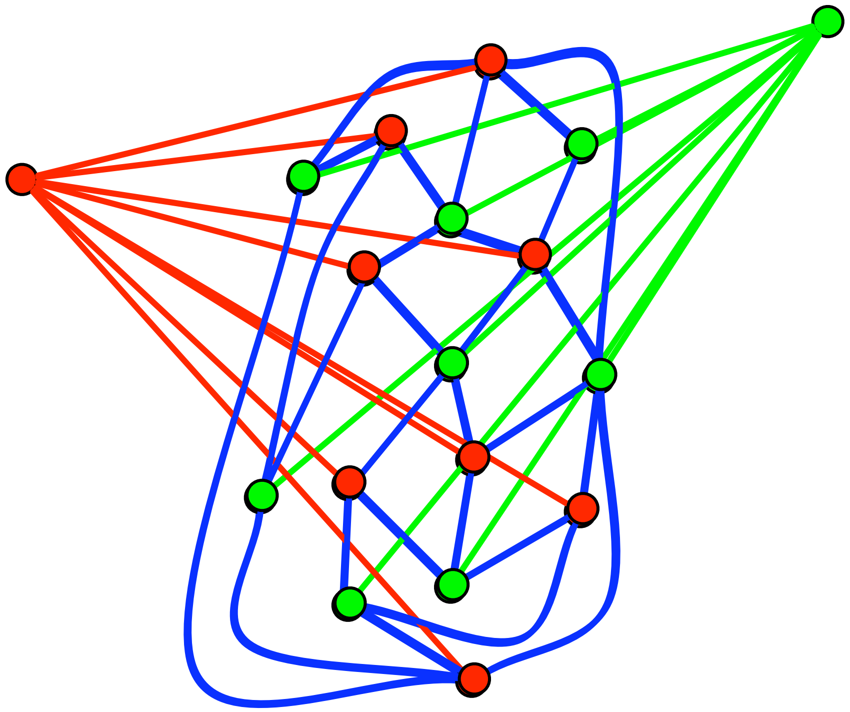
Reduction to Single Source and Sink



Reduction to Single Source and Sink



Reduction to Single Source and Sink



- reduction does not preserve planarity
- [Miller, Naor '91] - sources and sinks on a small number of faces

Known Results for Single Source/Sink

general graphs:

- $\tilde{O}(nm)$ - many results (blocking flow, push relabel)
- $O(m^{3/2} \log(n^2/m) \log U)$ - [Goldberg, Rao '97]

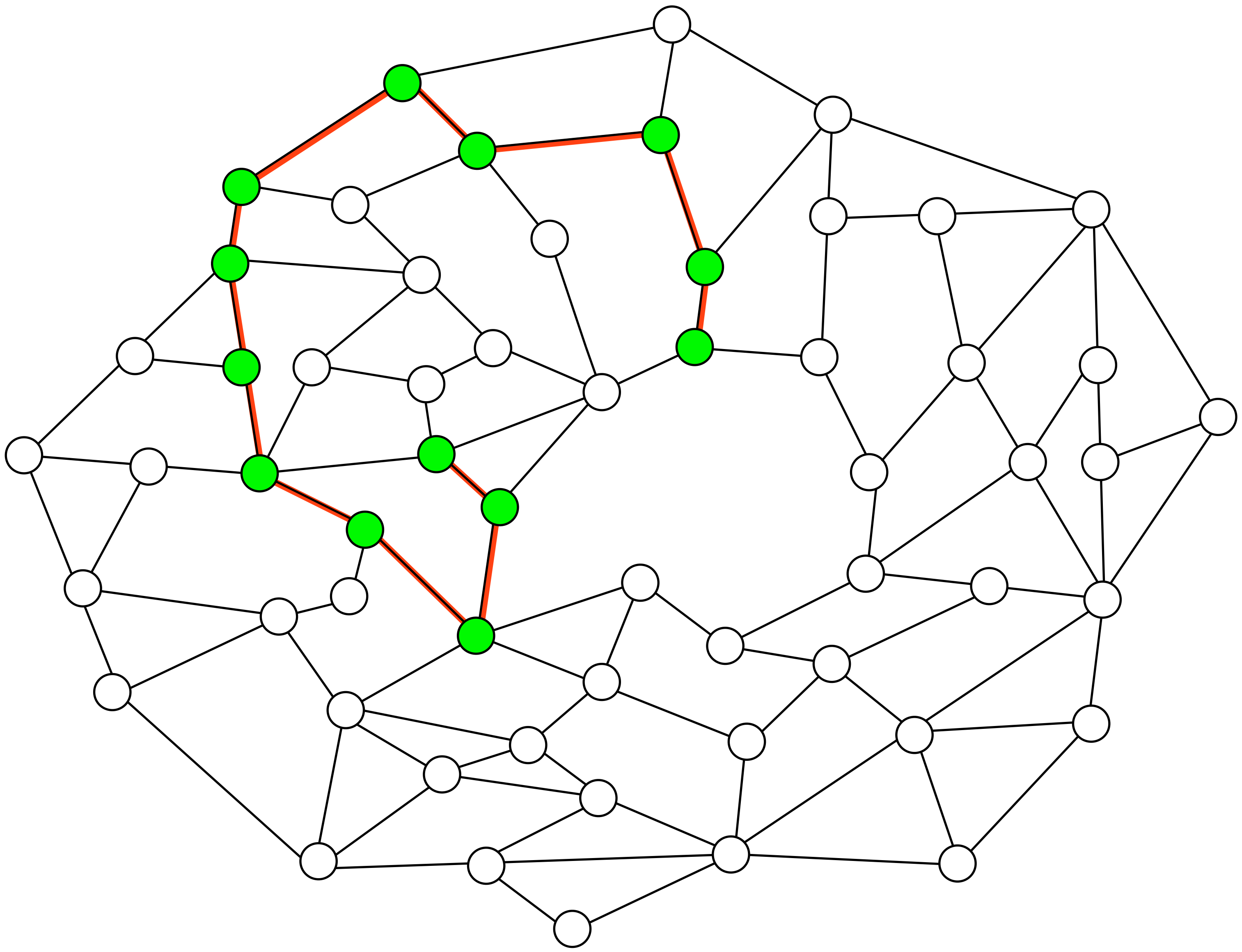
directed planar graphs:

- $O(n)$ - s and t on the same face [Hassin '81+ Henzinger et al. '94]
- $O(n \log n)$ [Borradaile, Klein '06]

Outline

- a few tools and definitions
- high-level description of recursive algorithm
- main ingredients for near-linear time

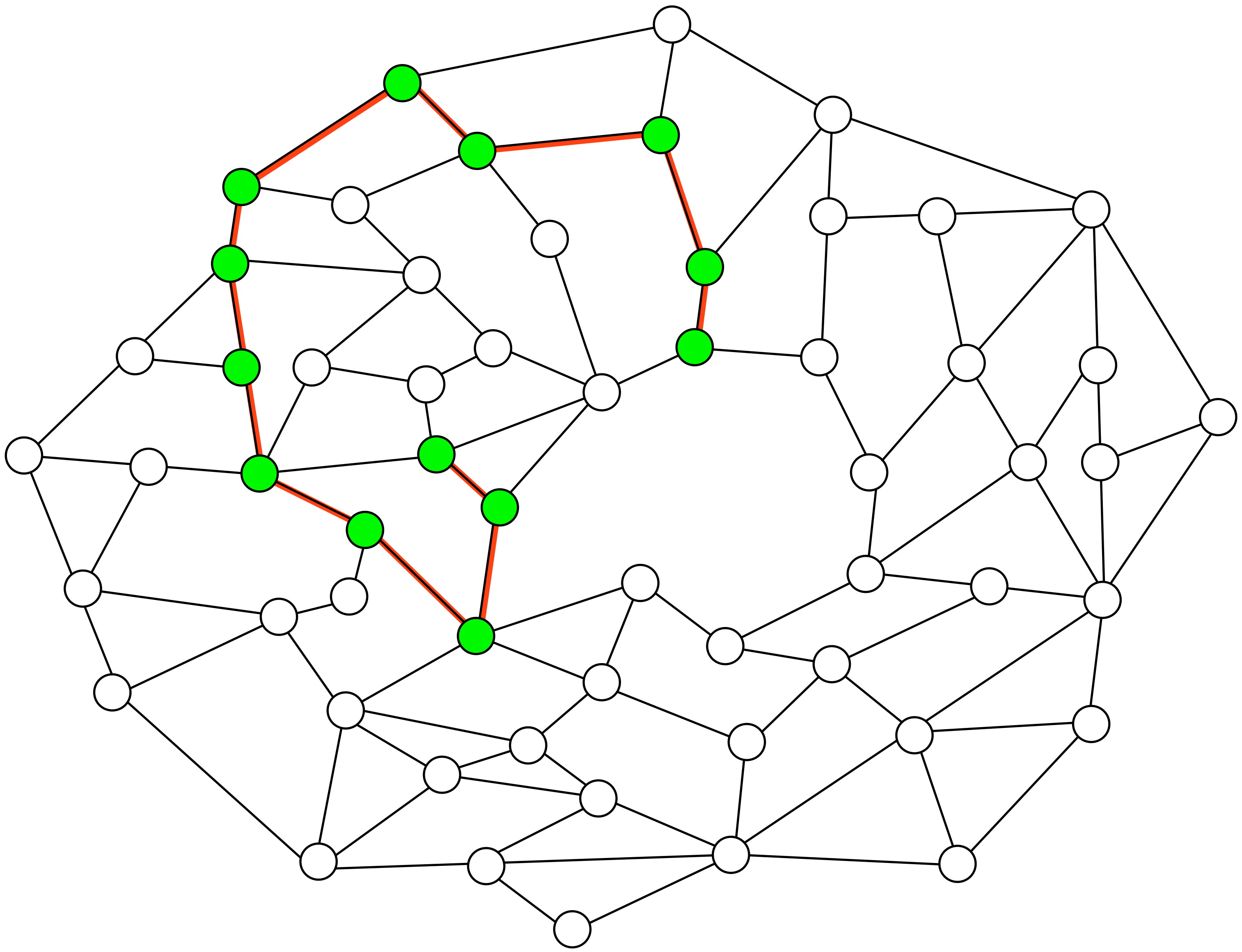
Multiple Sinks on a path



Multiple Sinks on a path

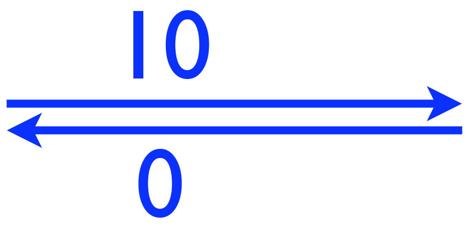
reduces to the single
sink case -
connect all sinks with
infinite-capacity edges

preserves planarity!



The Residual Graph

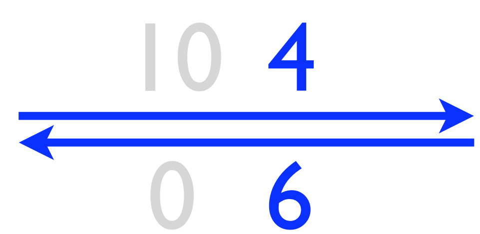
- given flow f in graph G with capacities $c(a)$,
  the residual graph G_f has same nodes and arcs as G
  and capacities $c_f(a) = c(a) - f(a)$



- a path P is residual if every arc of P has positive capacity

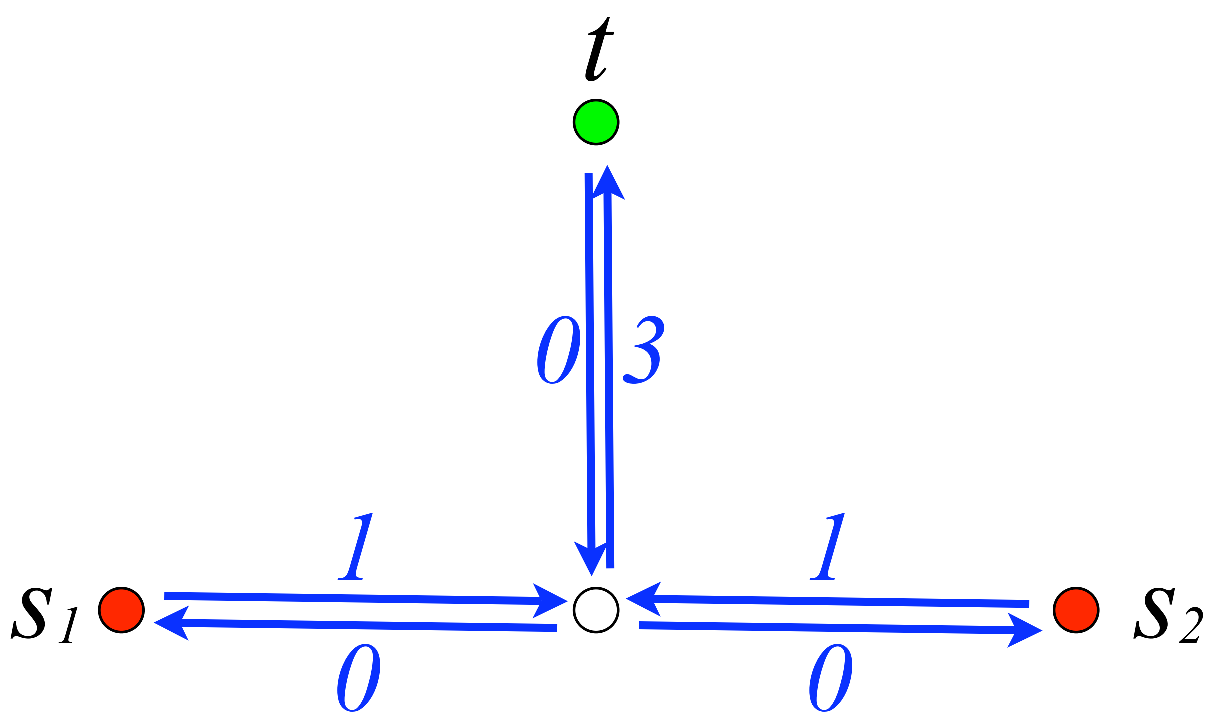
The Residual Graph

- given flow f in graph G with capacities $c(a)$, the residual graph G_f has same nodes and arcs as G and capacities $c_f(a) = c(a) - f(a)$

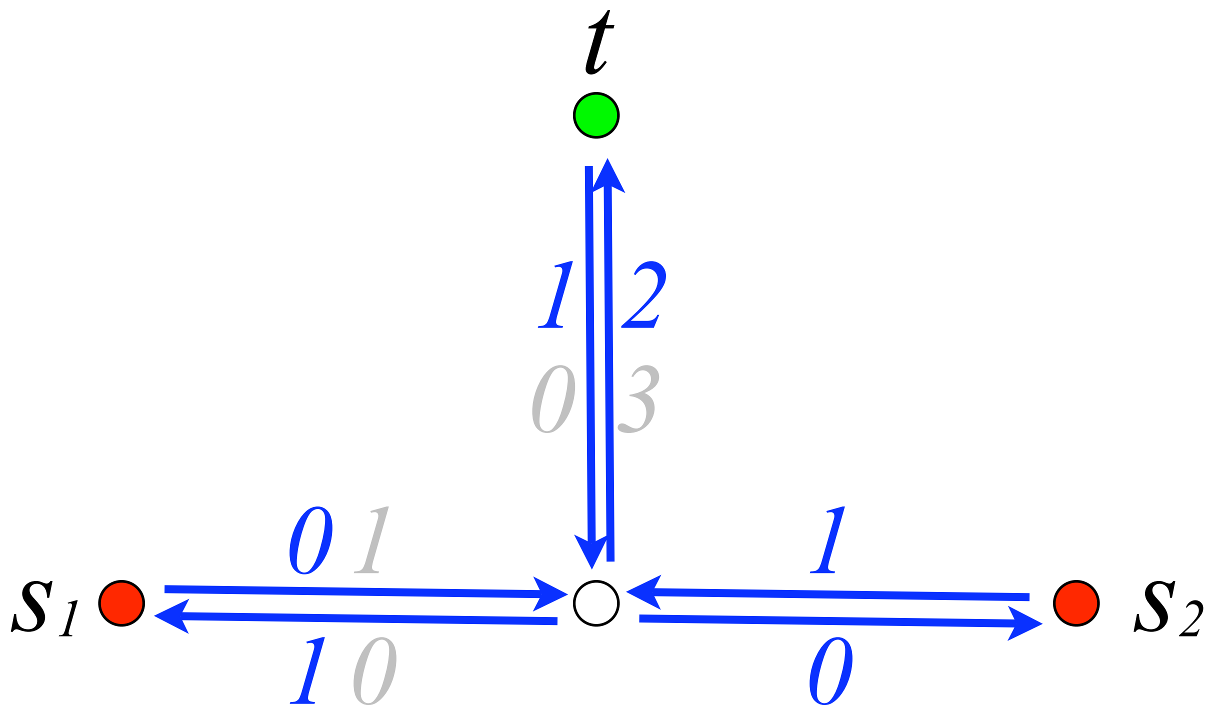


- a path P is residual if every arc of P has positive capacity

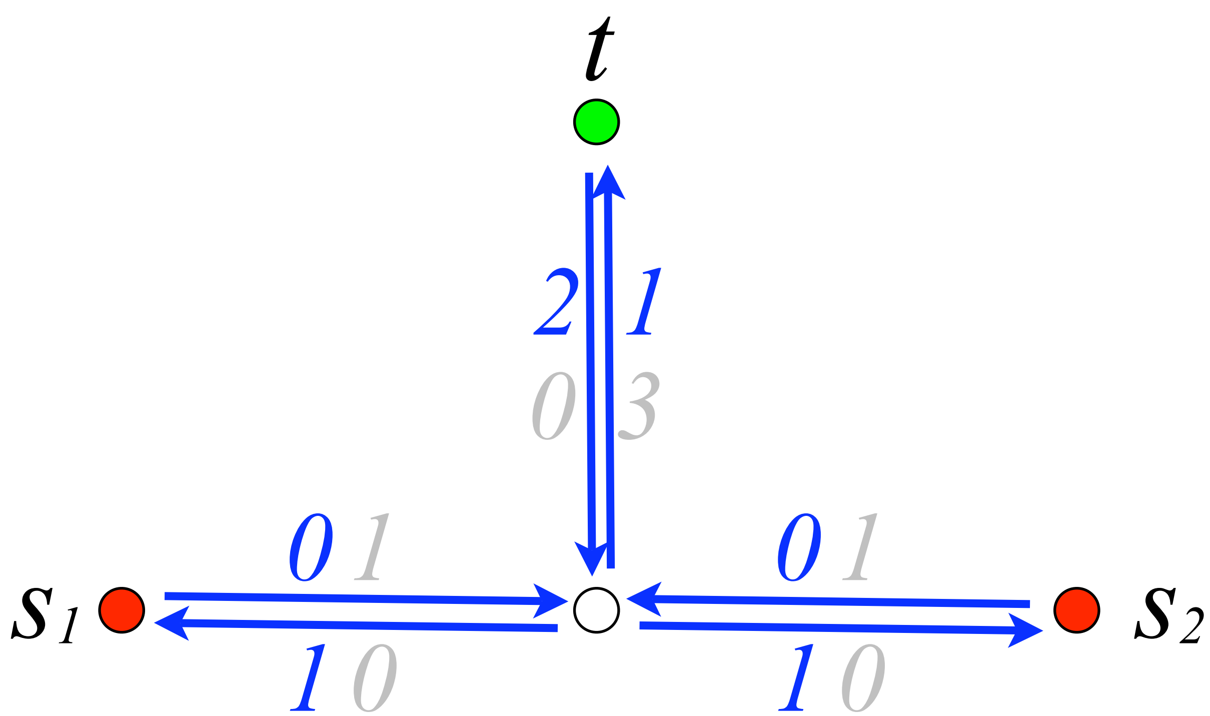
a flow f is maximum iff there are no residual paths from sources to sinks in G_f



a flow f is maximum iff there are no residual paths from sources to sinks in G_f



a flow f is maximum iff there are no residual paths from sources to sinks in G_f

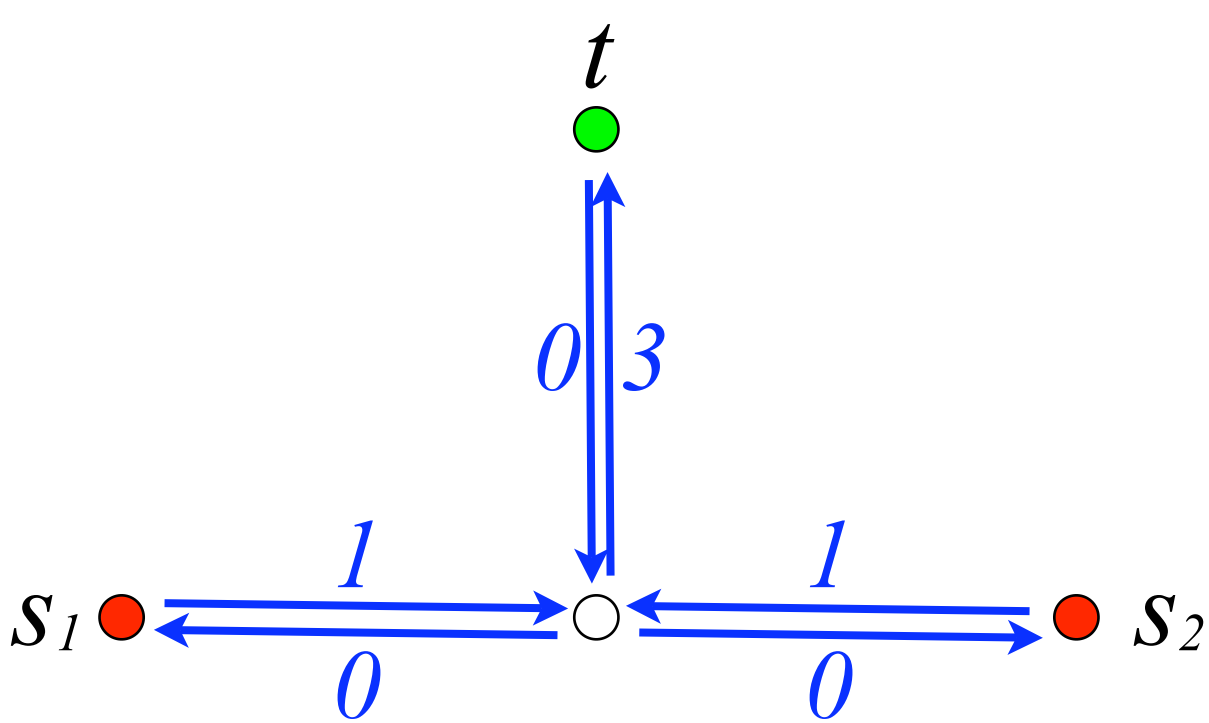


Flow Zoo

- excess flow at node v is the difference between amount of flow entering v and leaving v
  conservation ⇒ excess flow is zero
- pseudoflow: arc capacities are respected (conservation may not)
- feasible flow: pseudoflow that obeys conservation everywhere except sources and sinks
- circulation: pseudoflow that obeys conservation everywhere (even at sources and sinks)
- given a pseudoflow, it is possible to push back all positive/negative excess flow to/from its origin in linear time

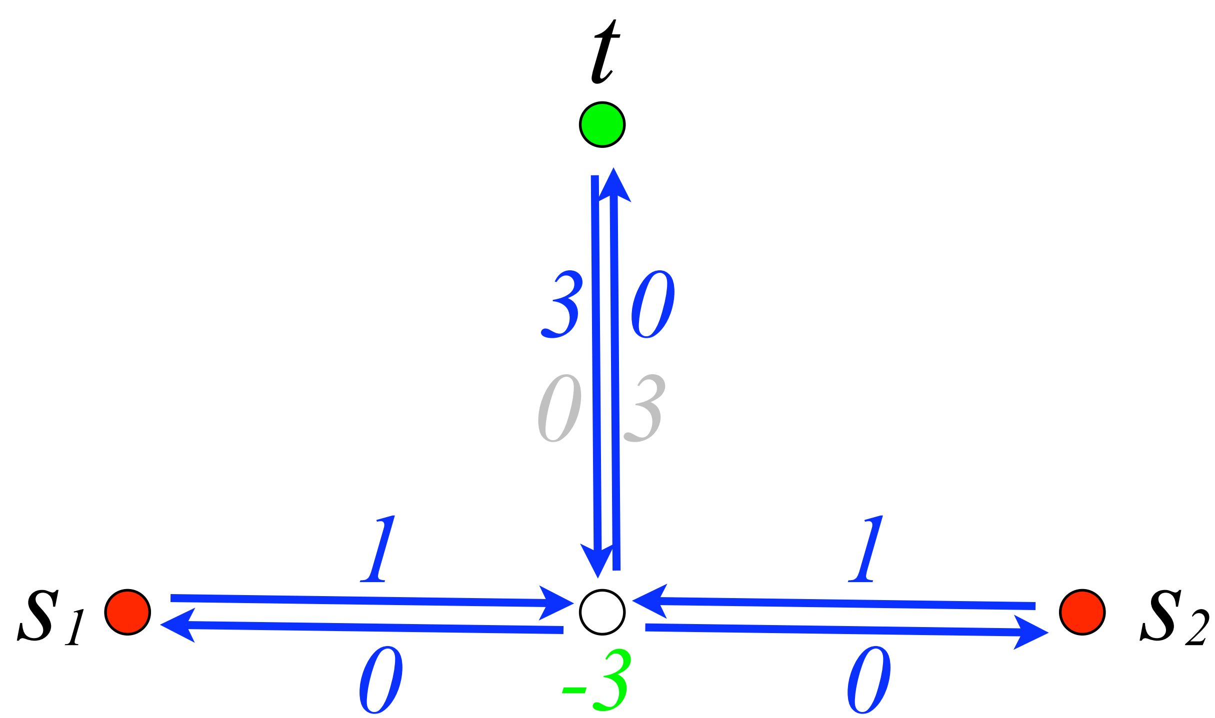
think of sources as having excess flow $+\infty$
think of sinks as having excess flow $-\infty$

a pseudoflow corresponds to a maximum flow iff
there are no residual paths from + to - in the residual graph



think of sources as having excess flow $+\infty$
think of sinks as having excess flow $-\infty$

a pseudoflow corresponds to a maximum flow iff
there are no residual paths from + to - in the residual graph

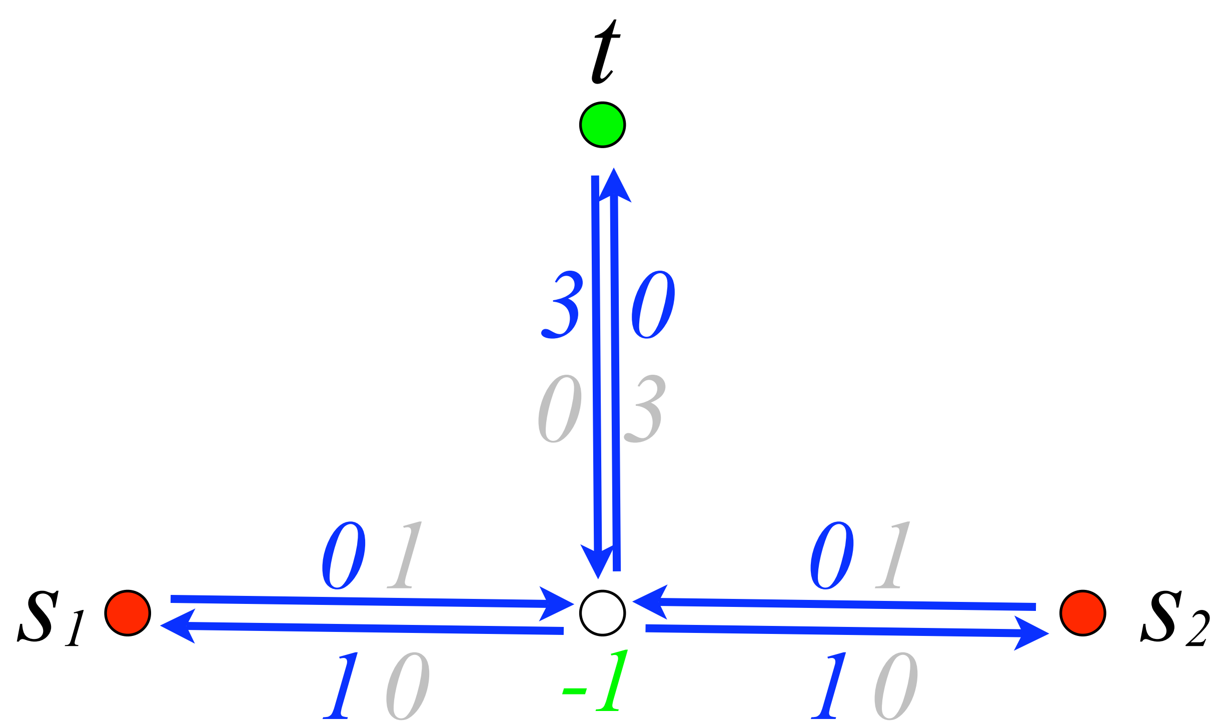


think of sources as having excess flow +∞

think of sinks as having excess flow -∞

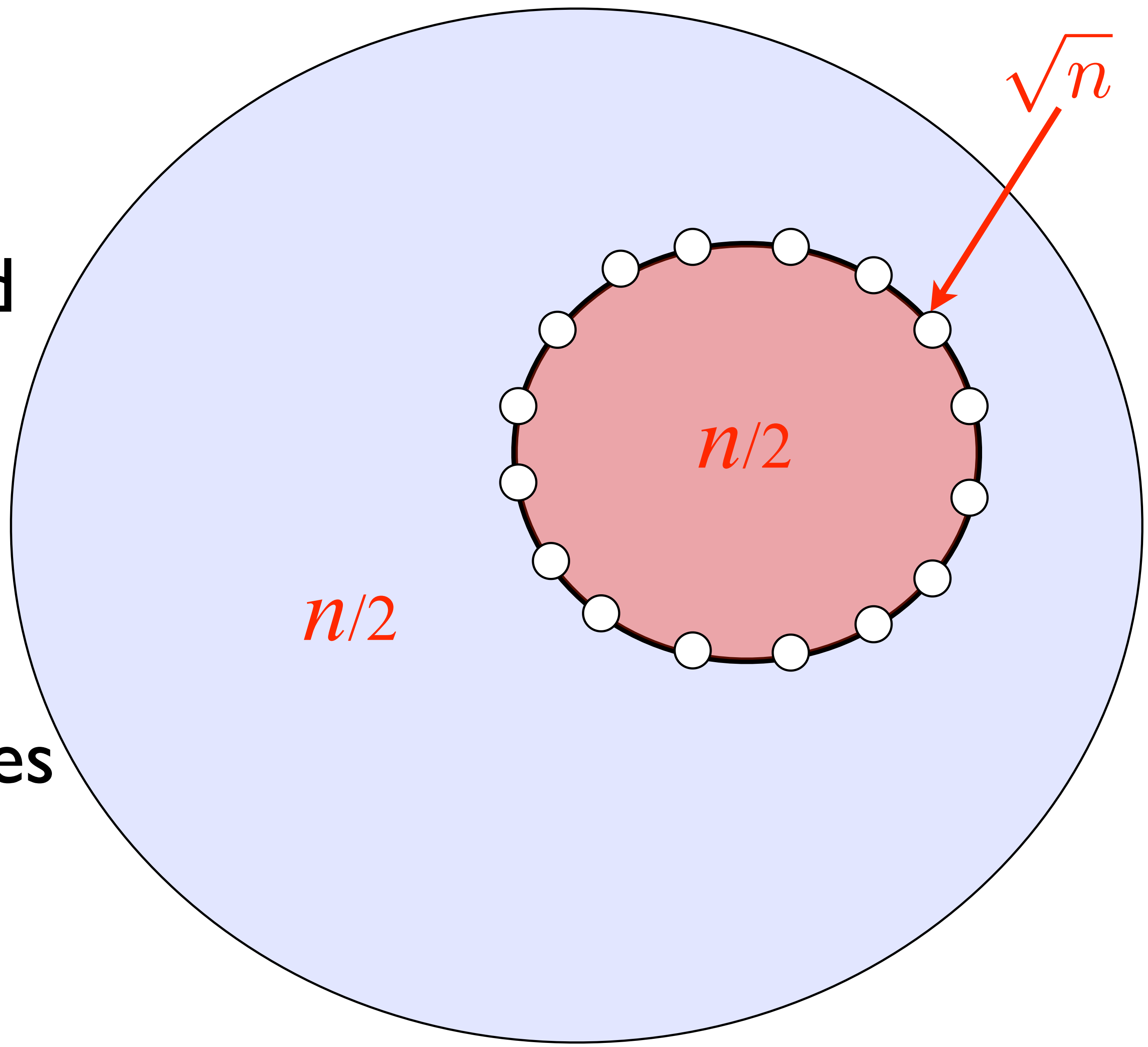
a pseudoflow corresponds to a maximum flow iff

there are no residual paths from + to - in the residual graph



Cycle Separators [Miller '86]

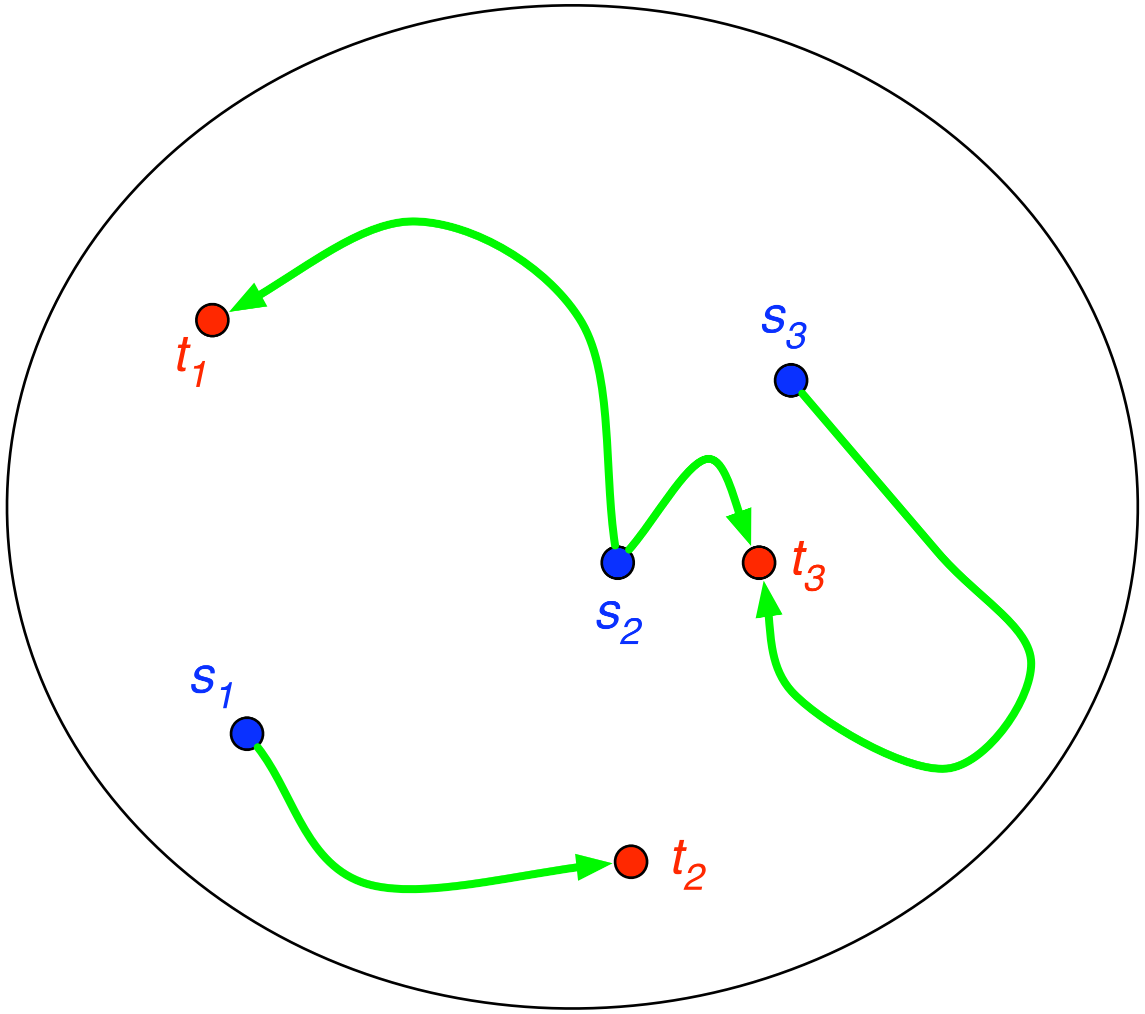
- simple cycle in a triangulated 2-connected planar graph
- balanced - between $n/3$ and $2n/3$ nodes on each side
- small: consists of $O(\sqrt{n})$ nodes
- can be found in $O(n)$ time



Outline

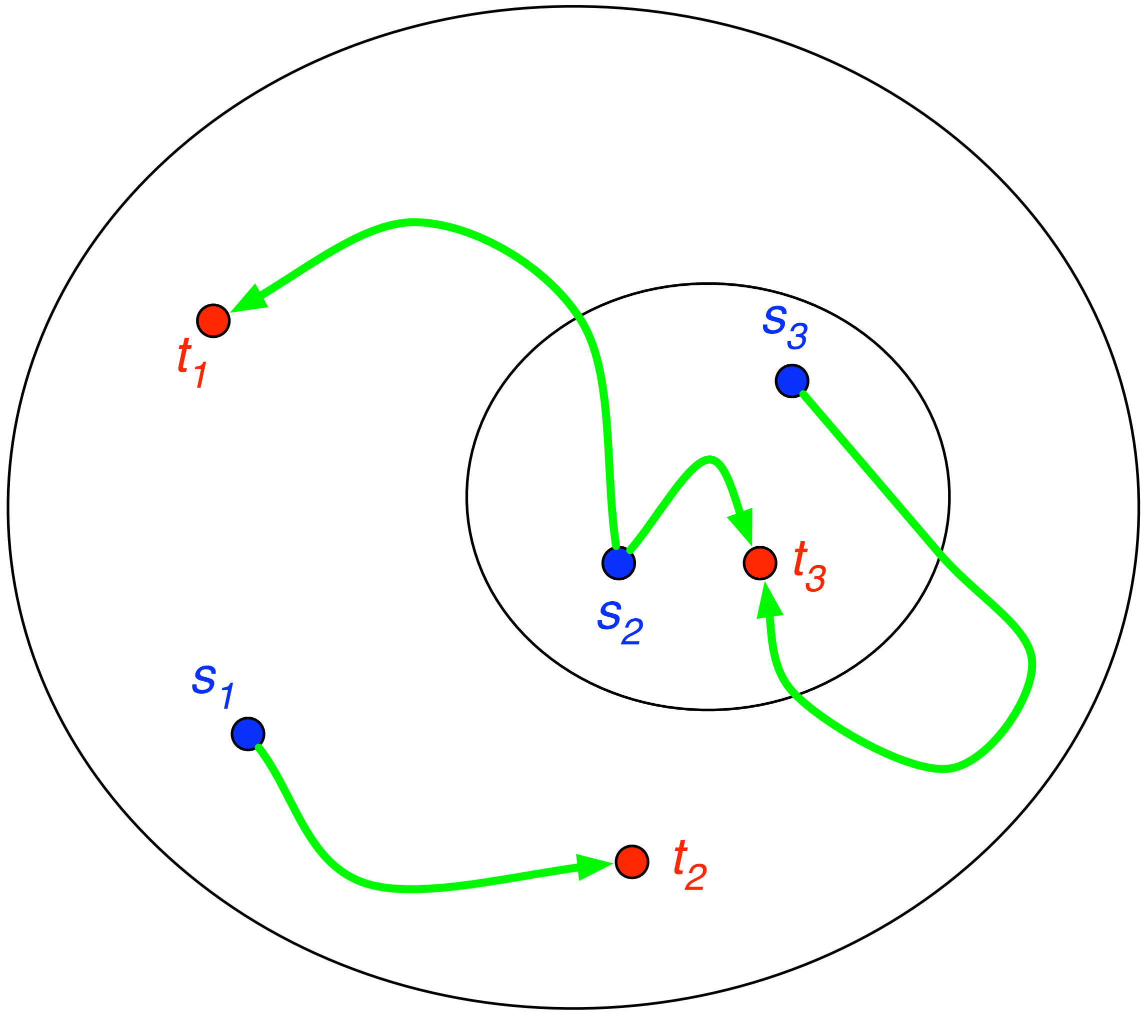
- a few tools and definitions
- high-level description of recursive algorithm
- main ingredients for near-linear time

Recursion, First try



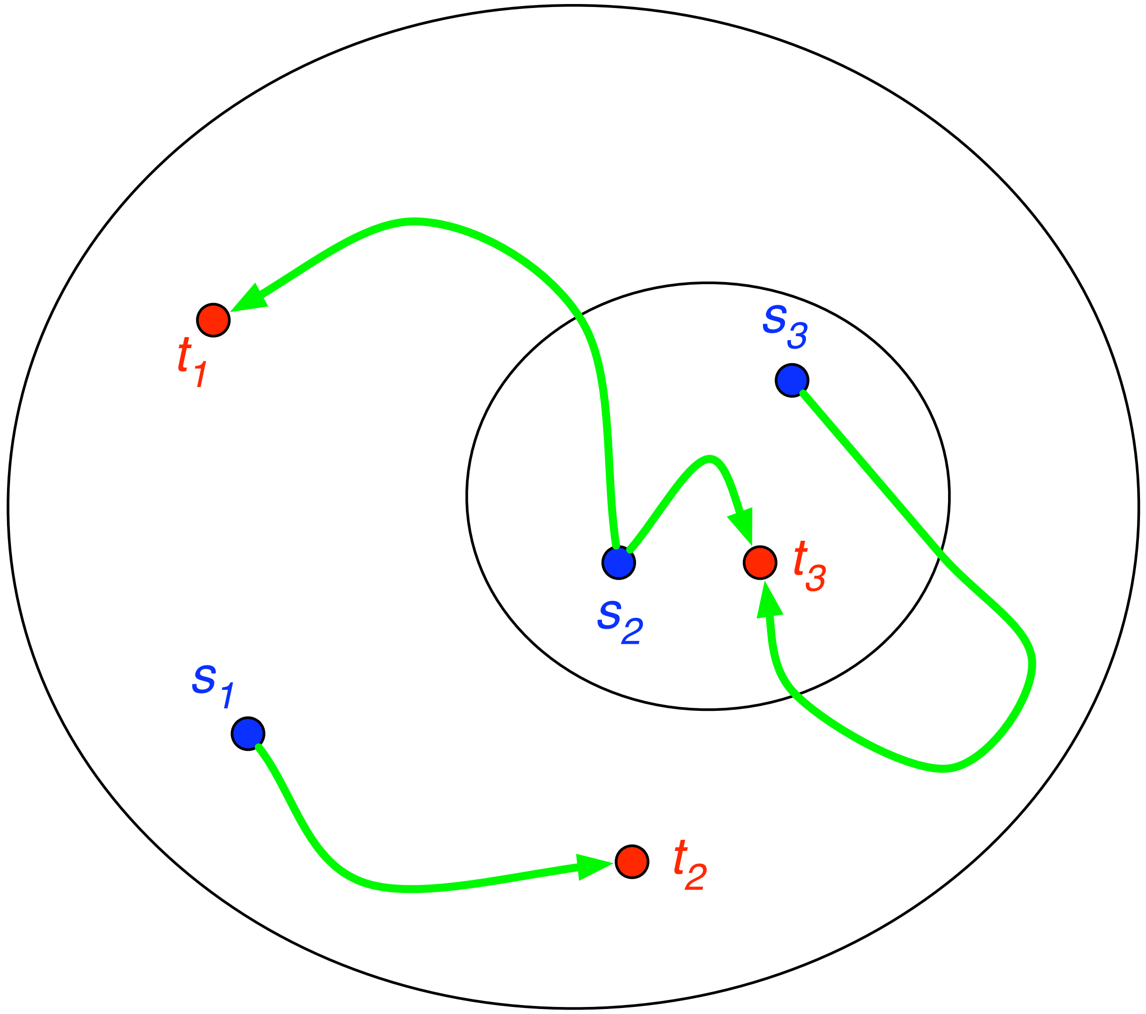
Recursion, First try

- find separator



Recursion, First try

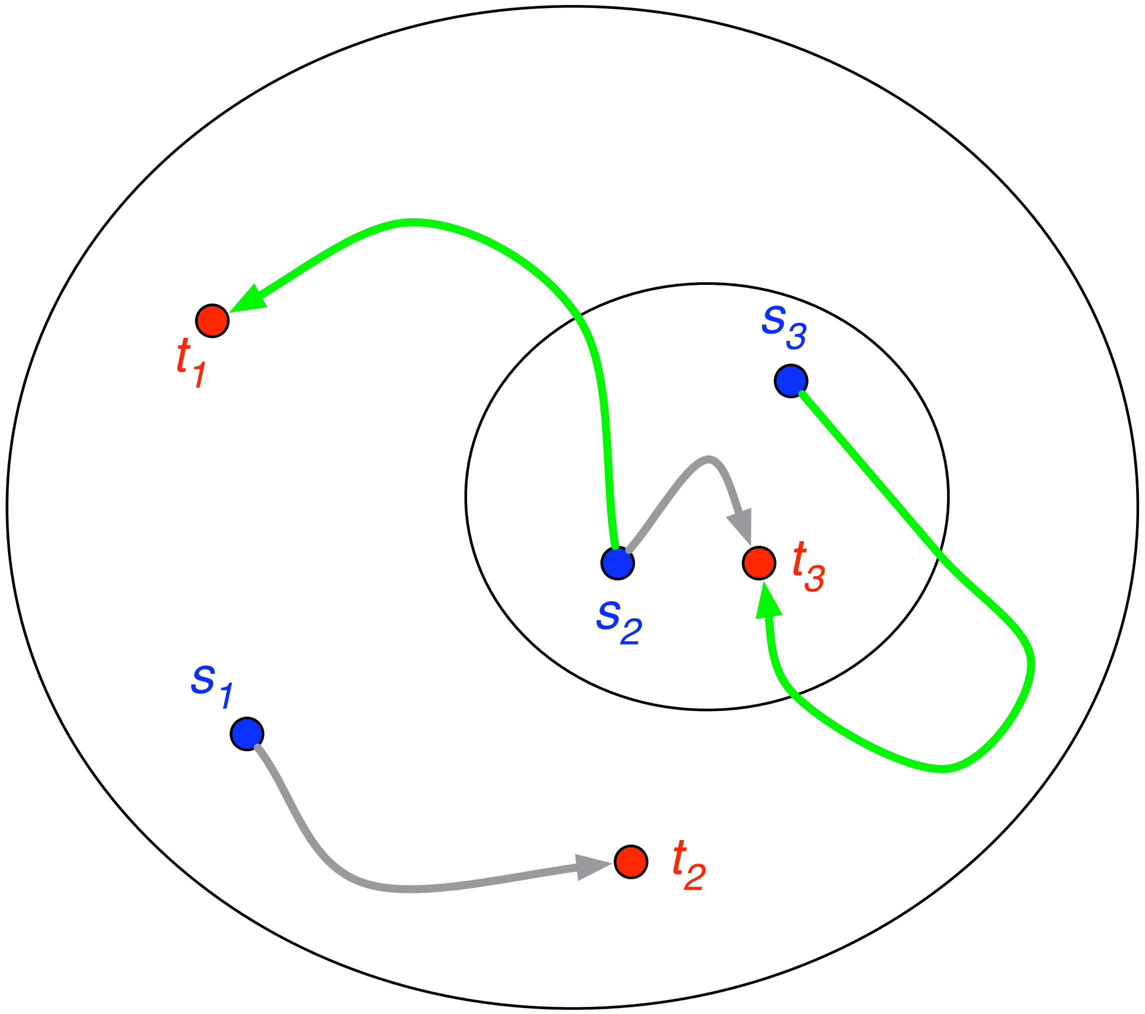
- find separator
- find maximum MSMS flow inside and outsider recursively



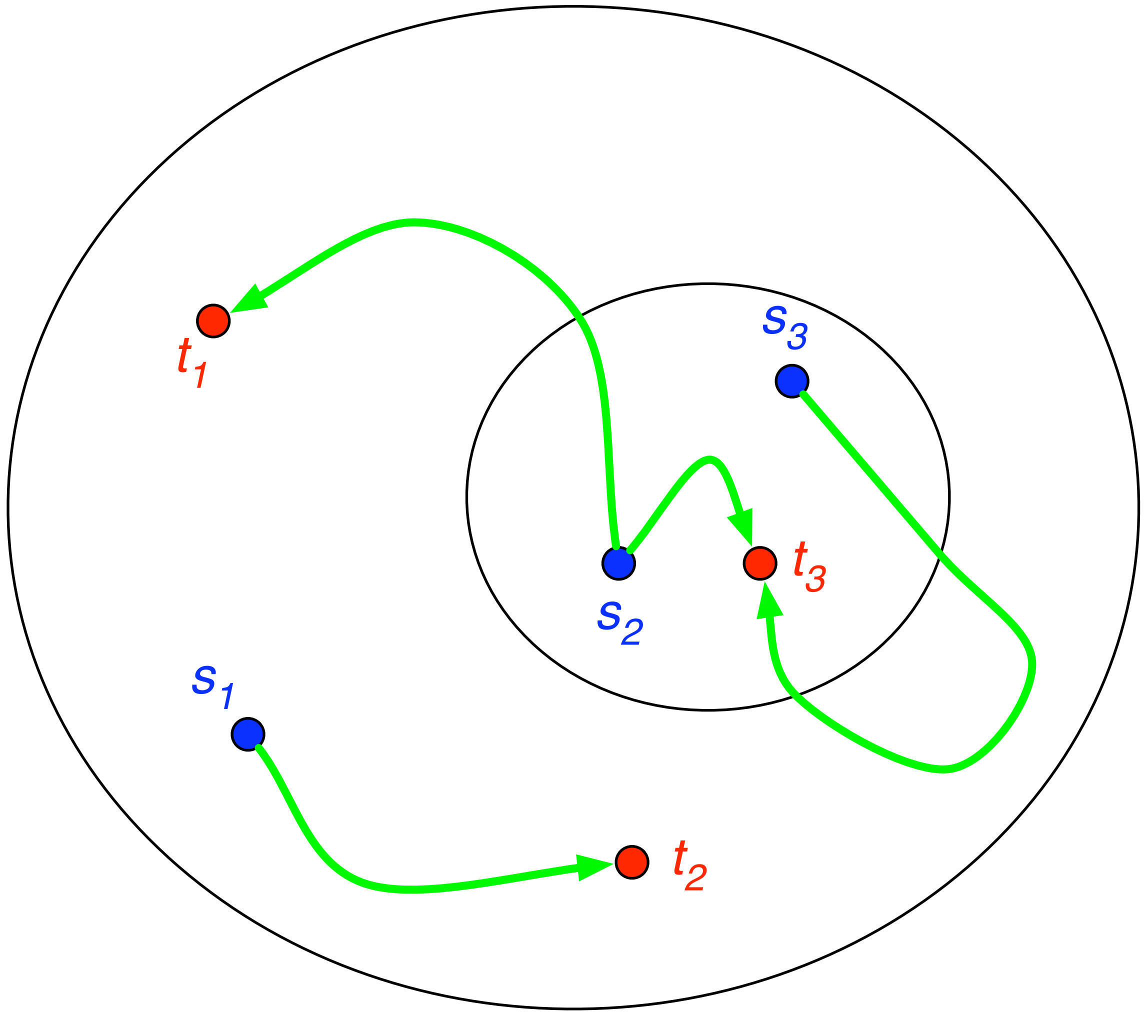
Recursion, First try

- find separator
- find maximum MSMS flow inside and outsider recursively

- no residual paths from sources to sinks in each subgraph

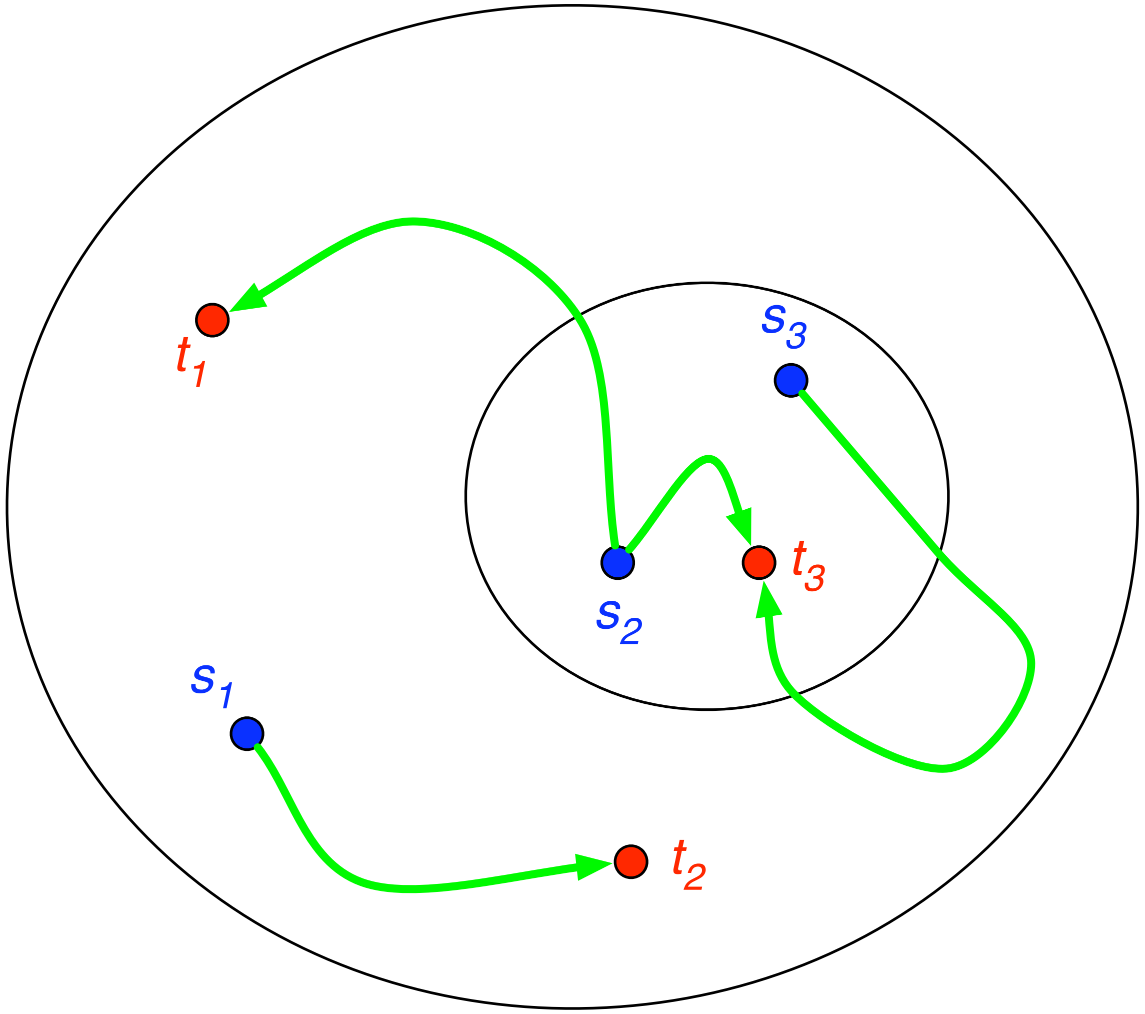


Recursion, Second try



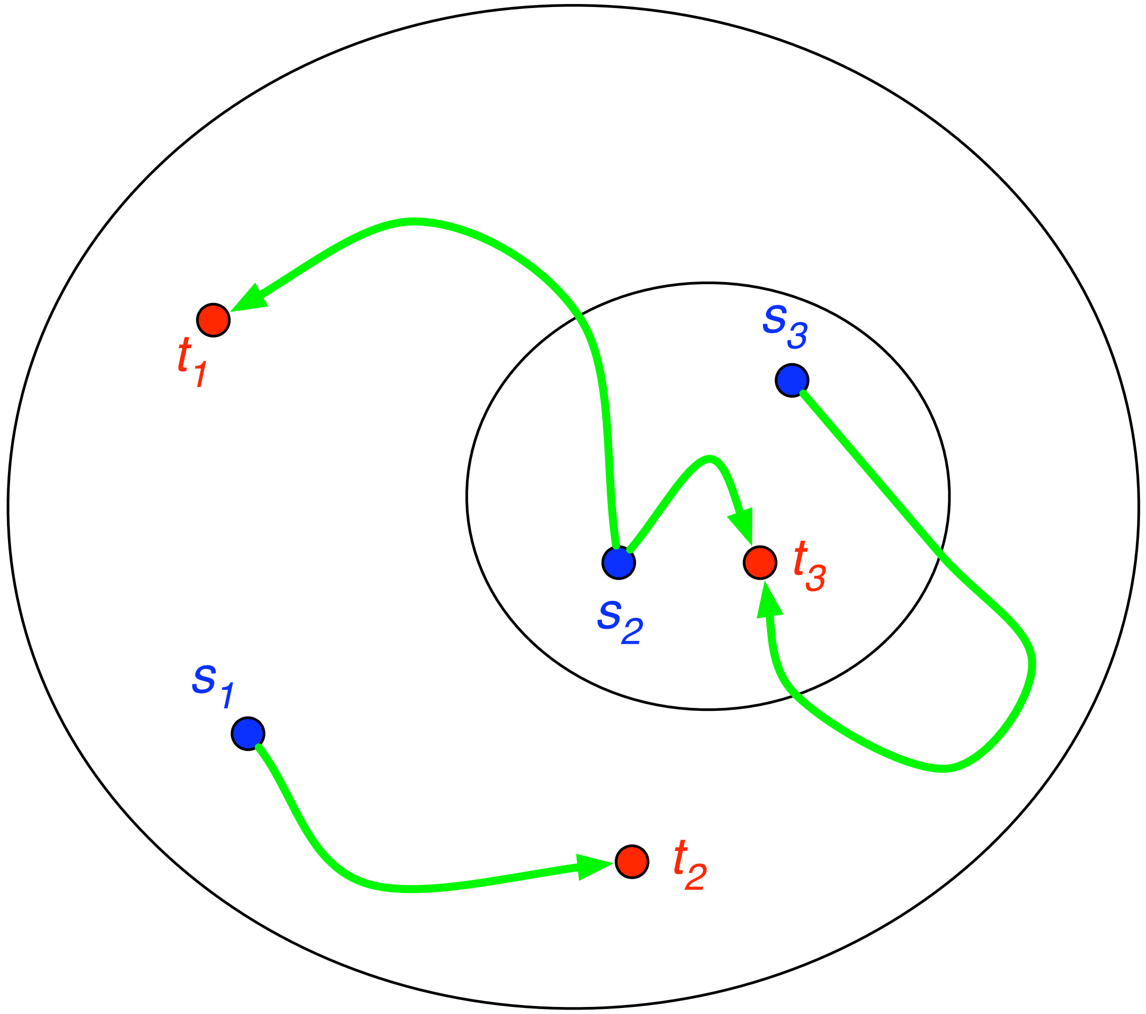
Recursion, Second try

- find separator



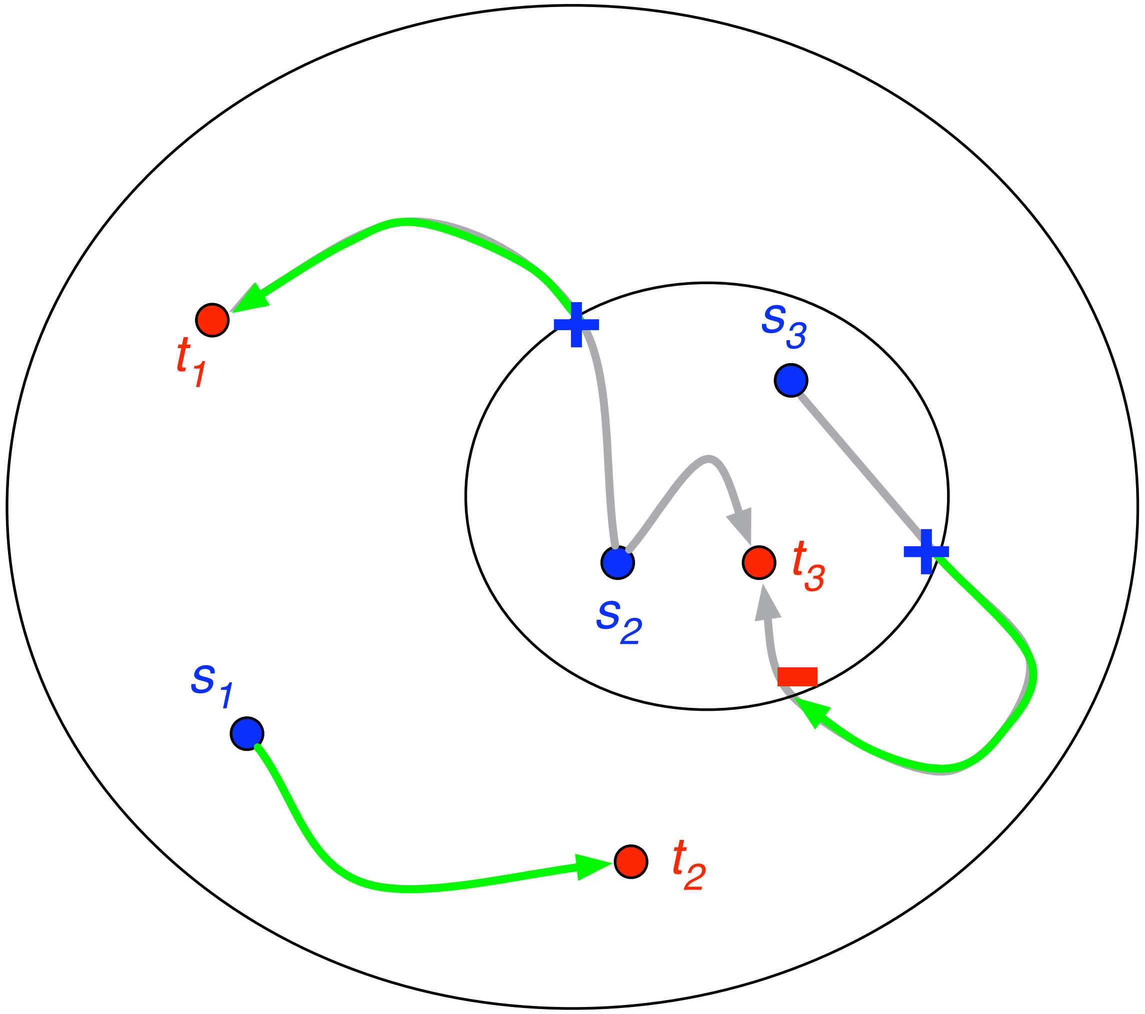
Recursion, Second try

- find separator
- recursive problem (almost): eliminate residual paths:
  - from sources to sinks
  - from sources to separator
  - from separator to sinks



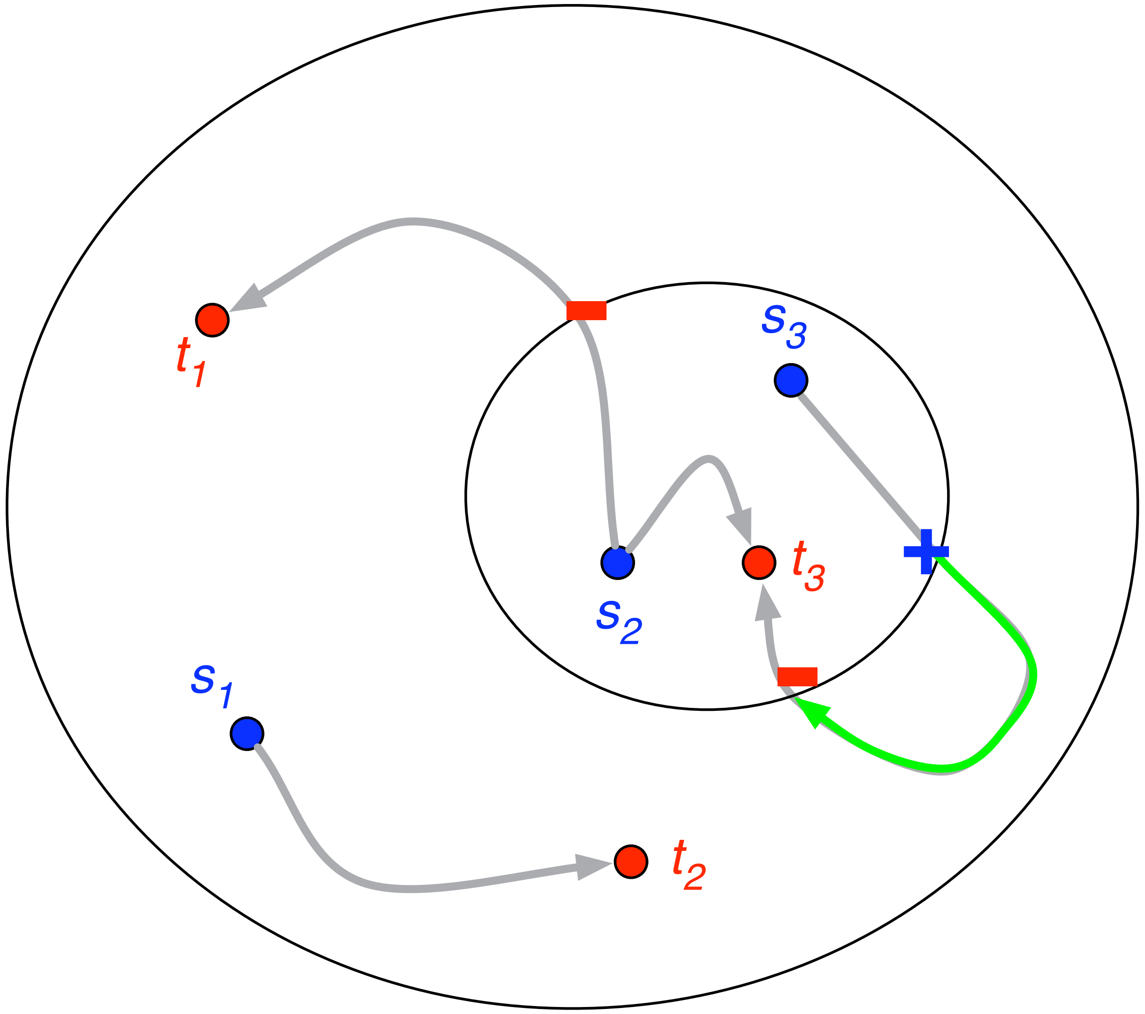
Recursion, Second try

- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks



Recursion, Second try

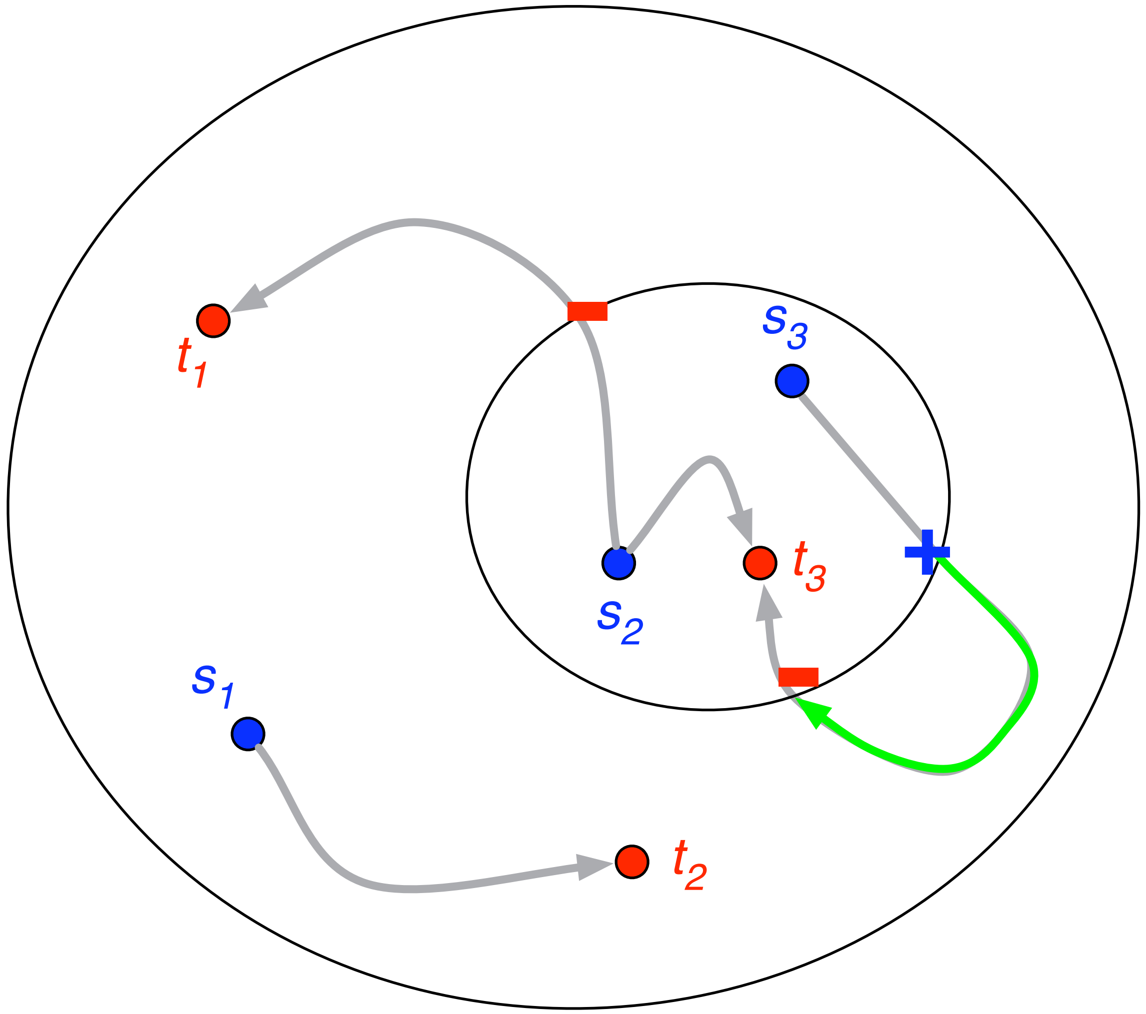
- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks



Recursion, Second try

- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks

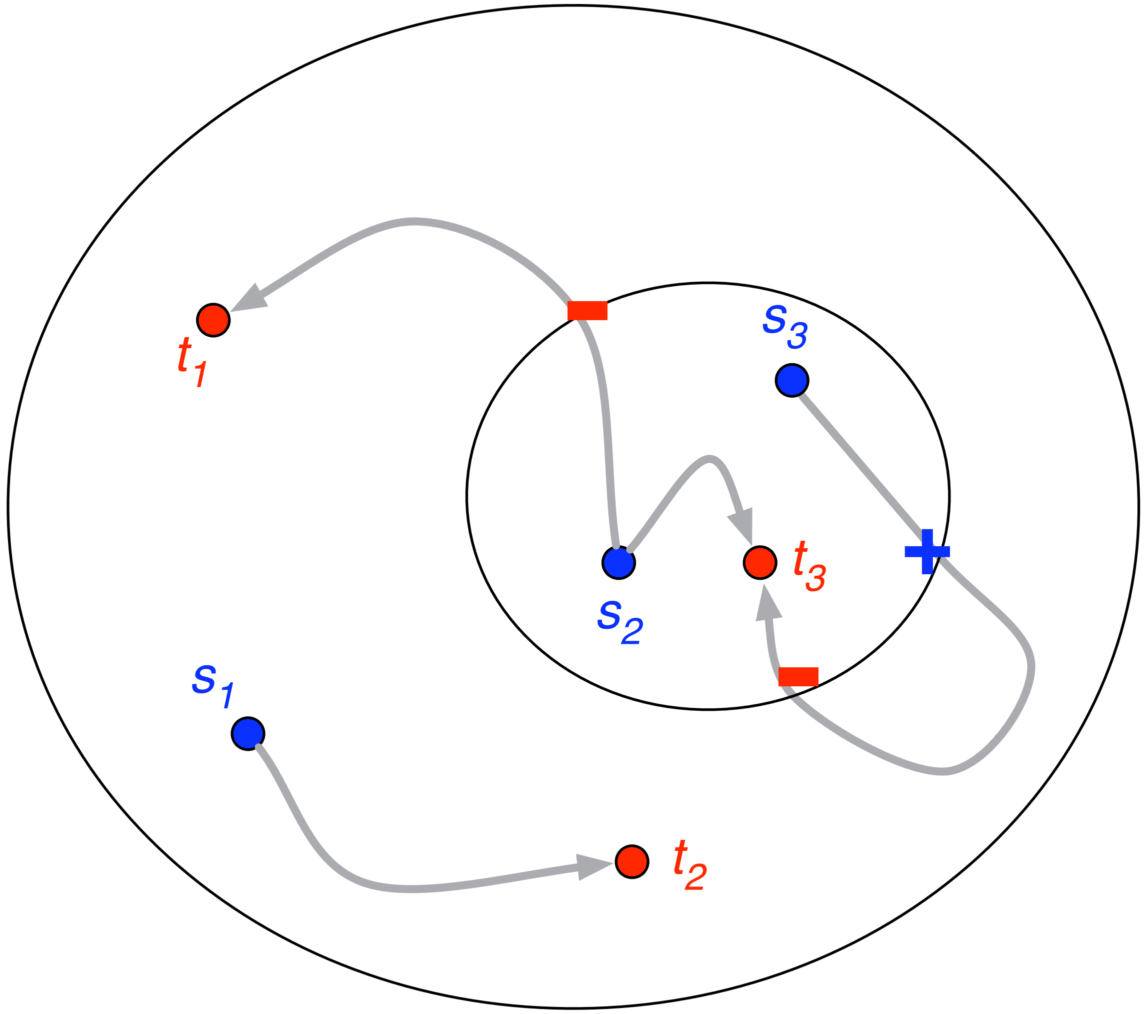
- eliminate residual paths from + to - on separator



Recursion, Second try

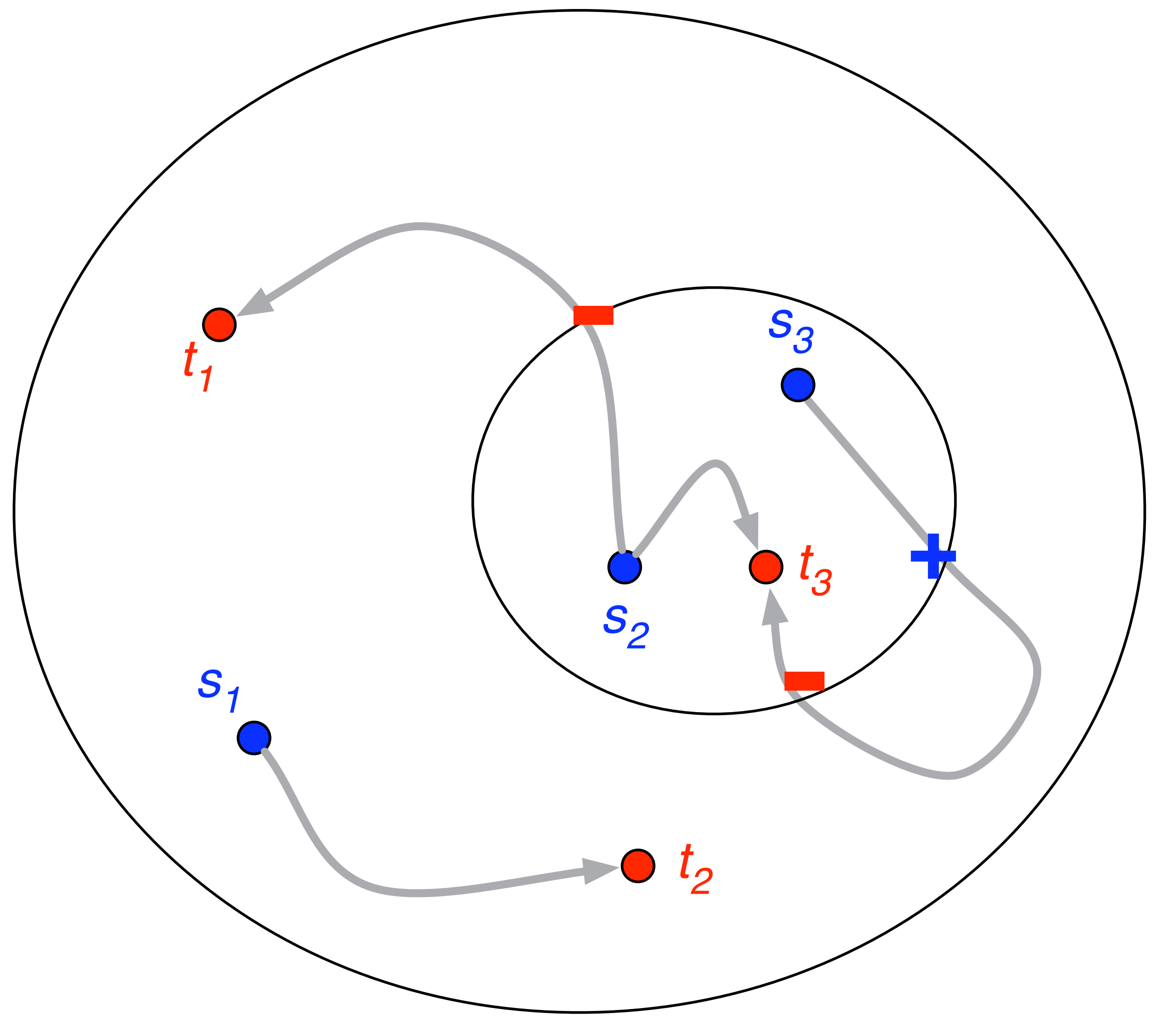
- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks

- eliminate residual paths from + to - on separator



Recursion, Second try

- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks



- eliminate residual paths from + to - on separator

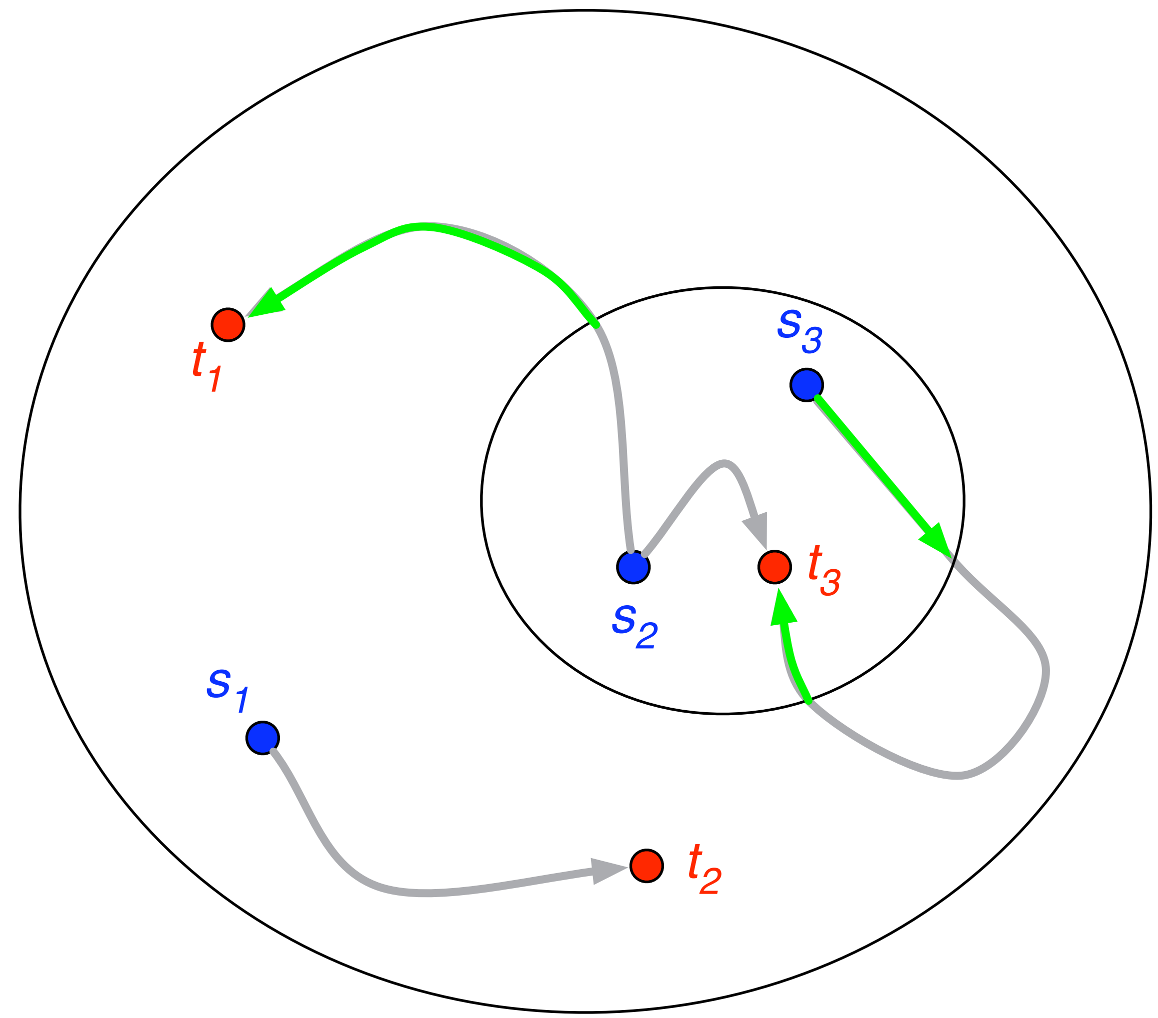
- return flow from + to sources and from sinks to -

Recursion, Second try

- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks

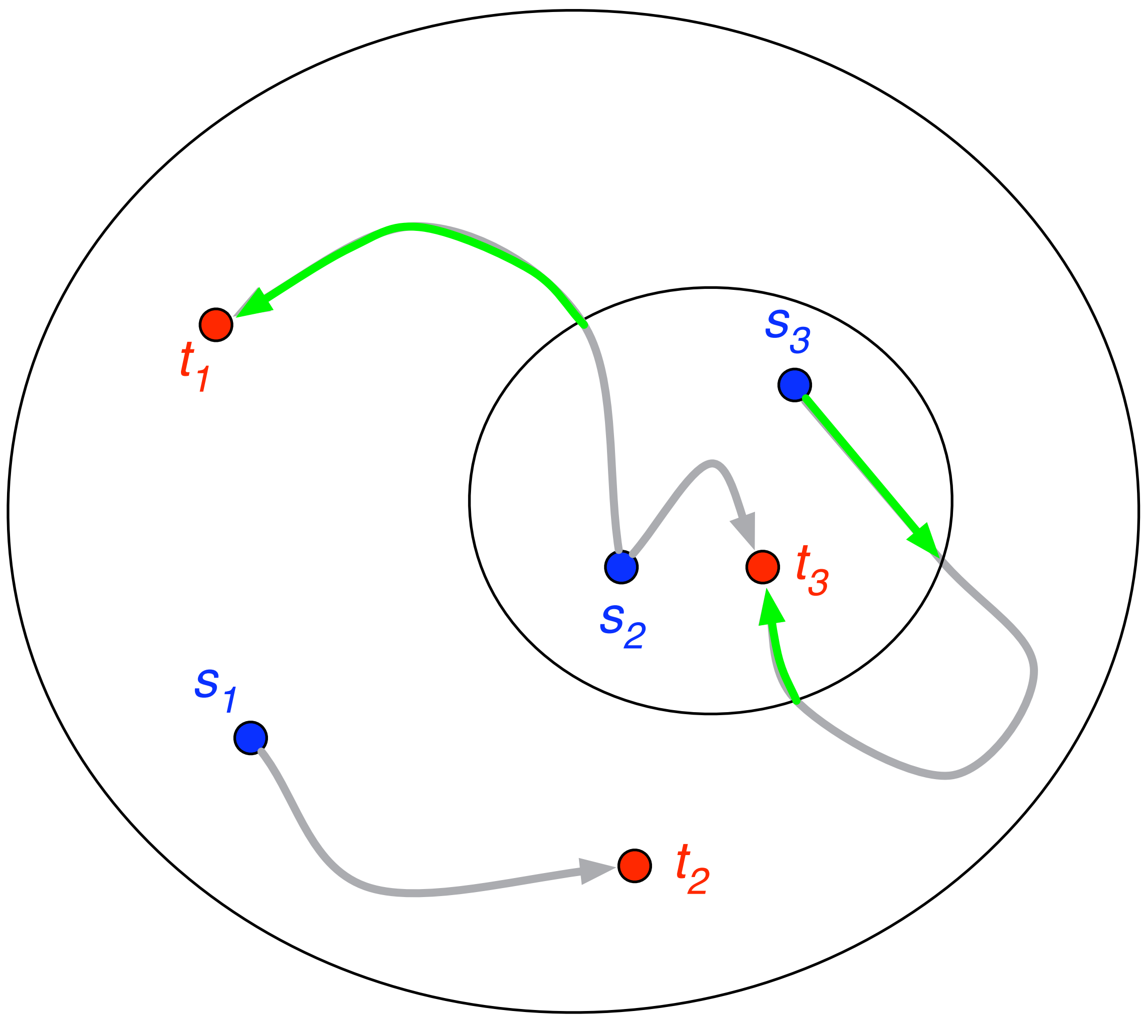
- eliminate residual paths from + to - on separator

- return flow from + to sources and from sinks to -



Recursion, Second try

- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks

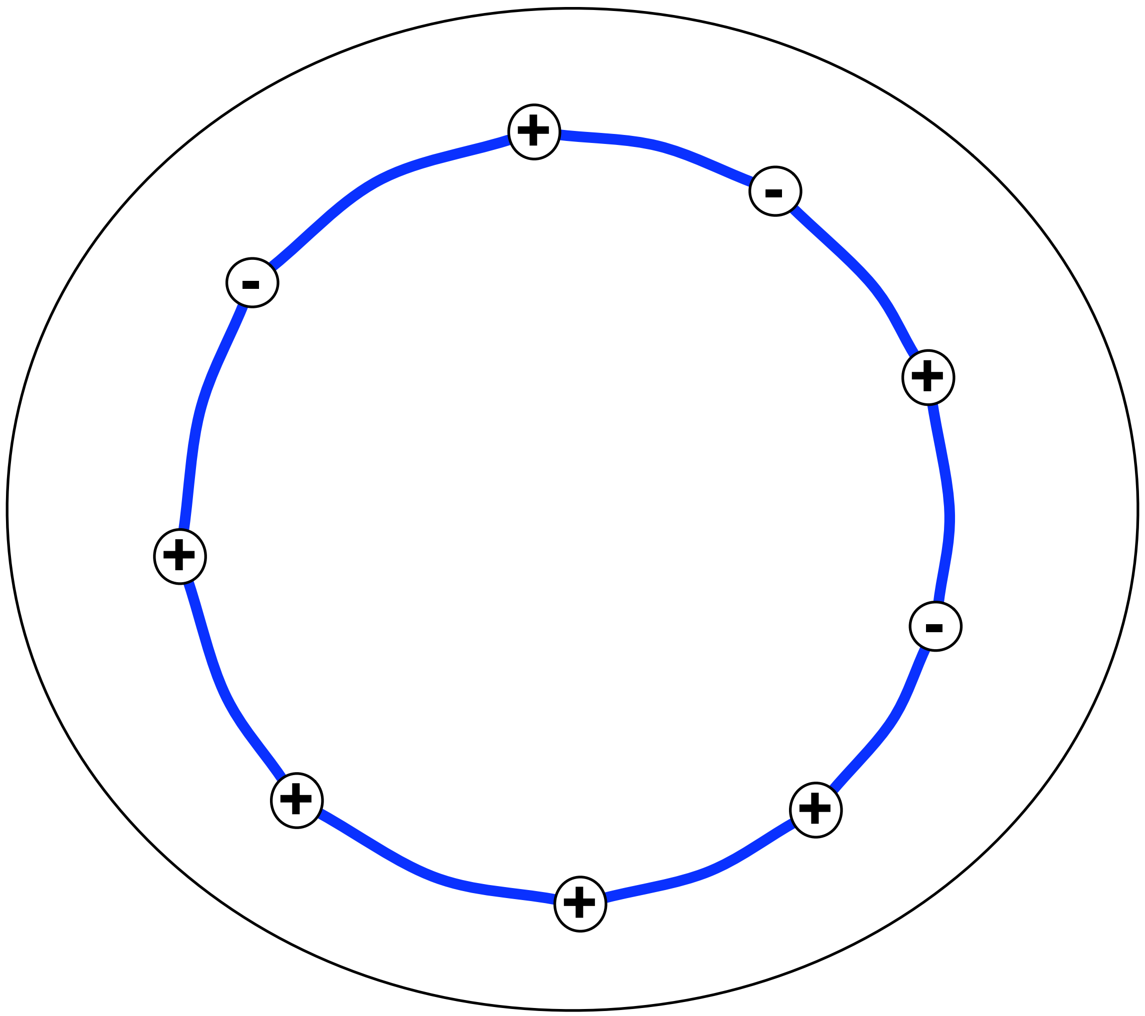


- eliminate residual paths from + to - on separator

- return flow from + to sources and from sinks to -

Fixing the Separator

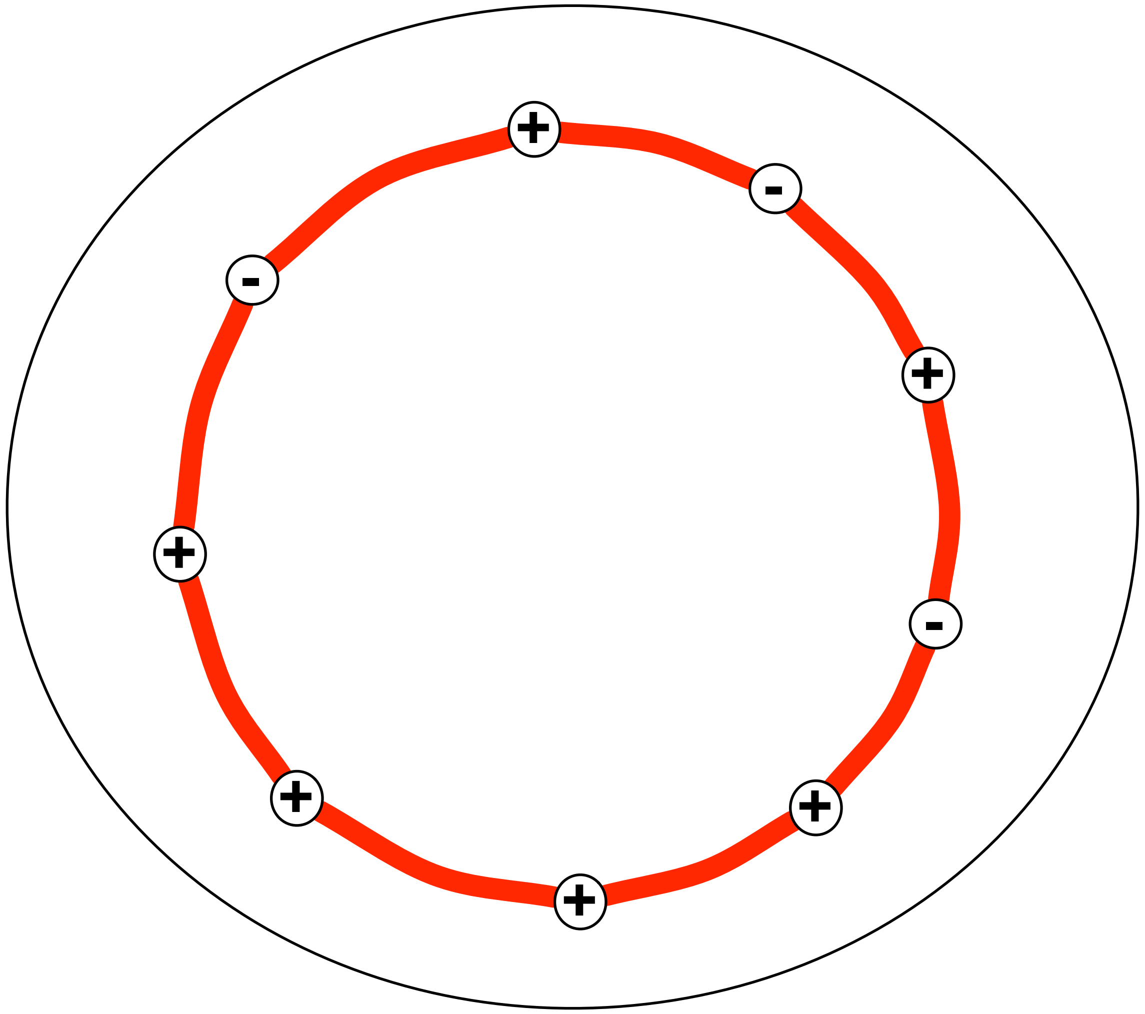
eliminate residual paths from + to - on separator



Fixing the Separator

eliminate residual paths from + to - on separator

- make capacity of separator edges infinite
- handle nodes one by one:

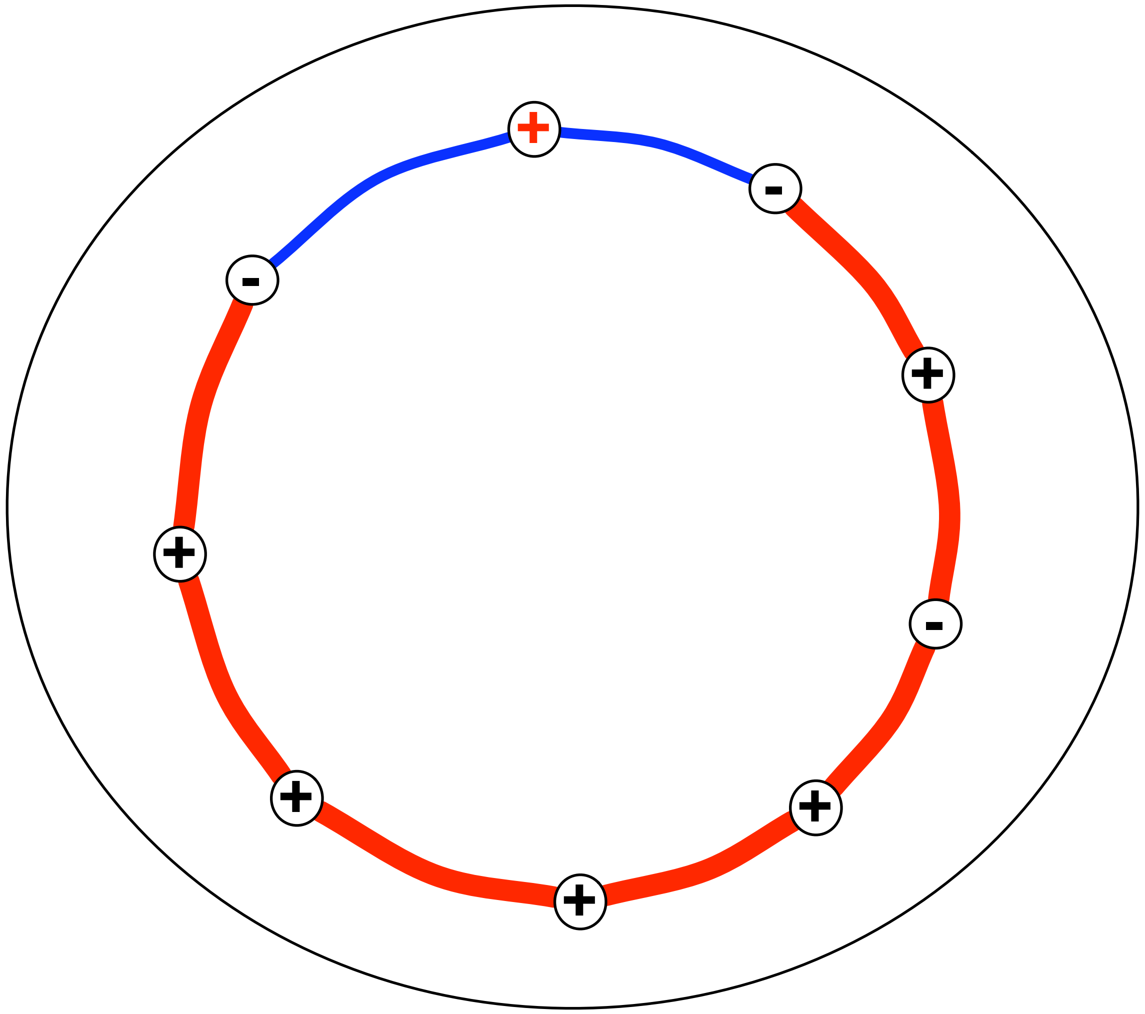


Fixing the Separator

eliminate residual paths from + to - on separator

- make capacity of separator edges infinite
- handle nodes one by one:

- reduce capacity of incident edges back to original

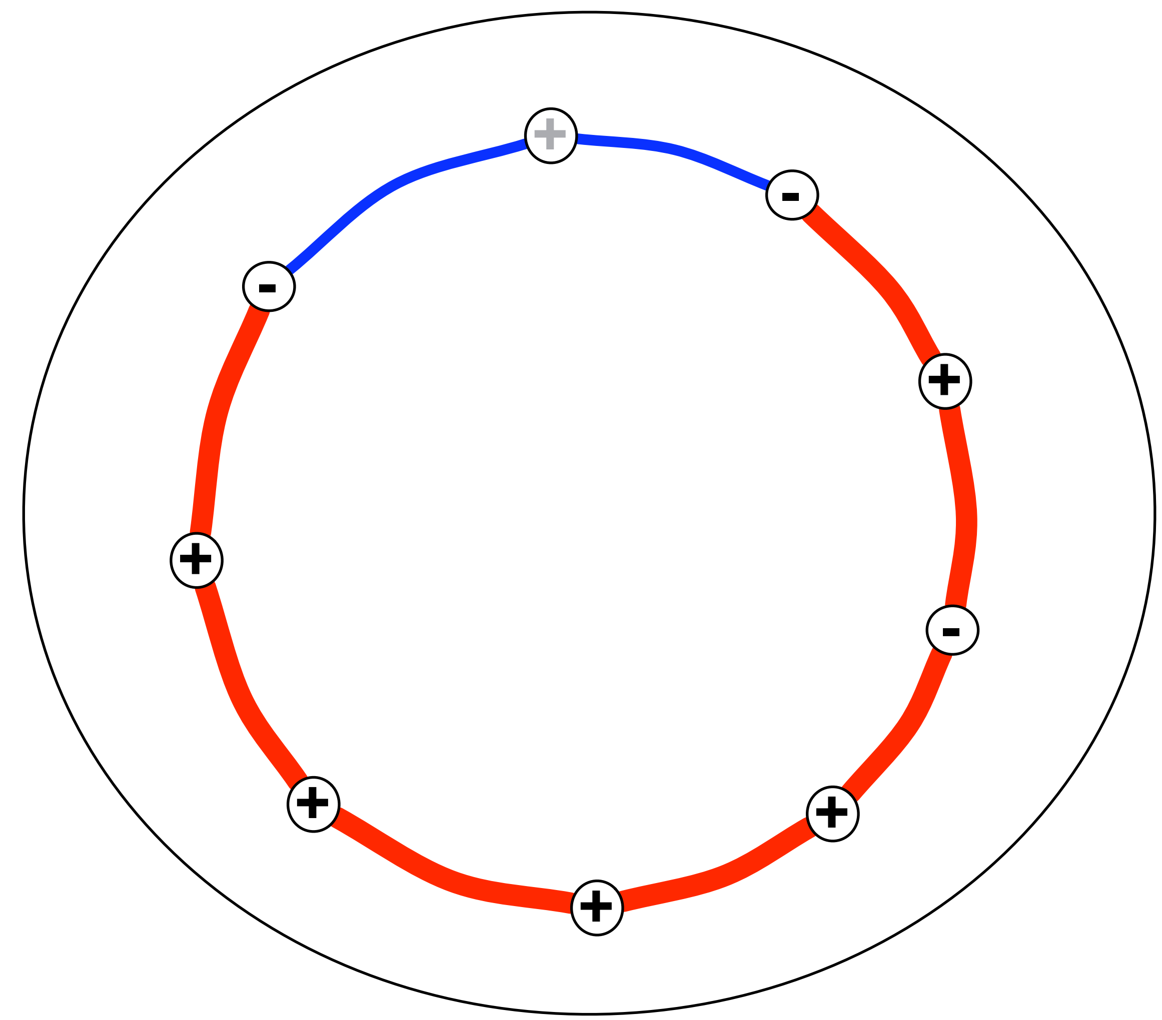


Fixing the Separator

eliminate residual paths from + to - on separator

- make capacity of separator edges infinite
- handle nodes one by one:

- reduce capacity of incident edges back to original
- push + excess to neighbor using max-flow in residual graph

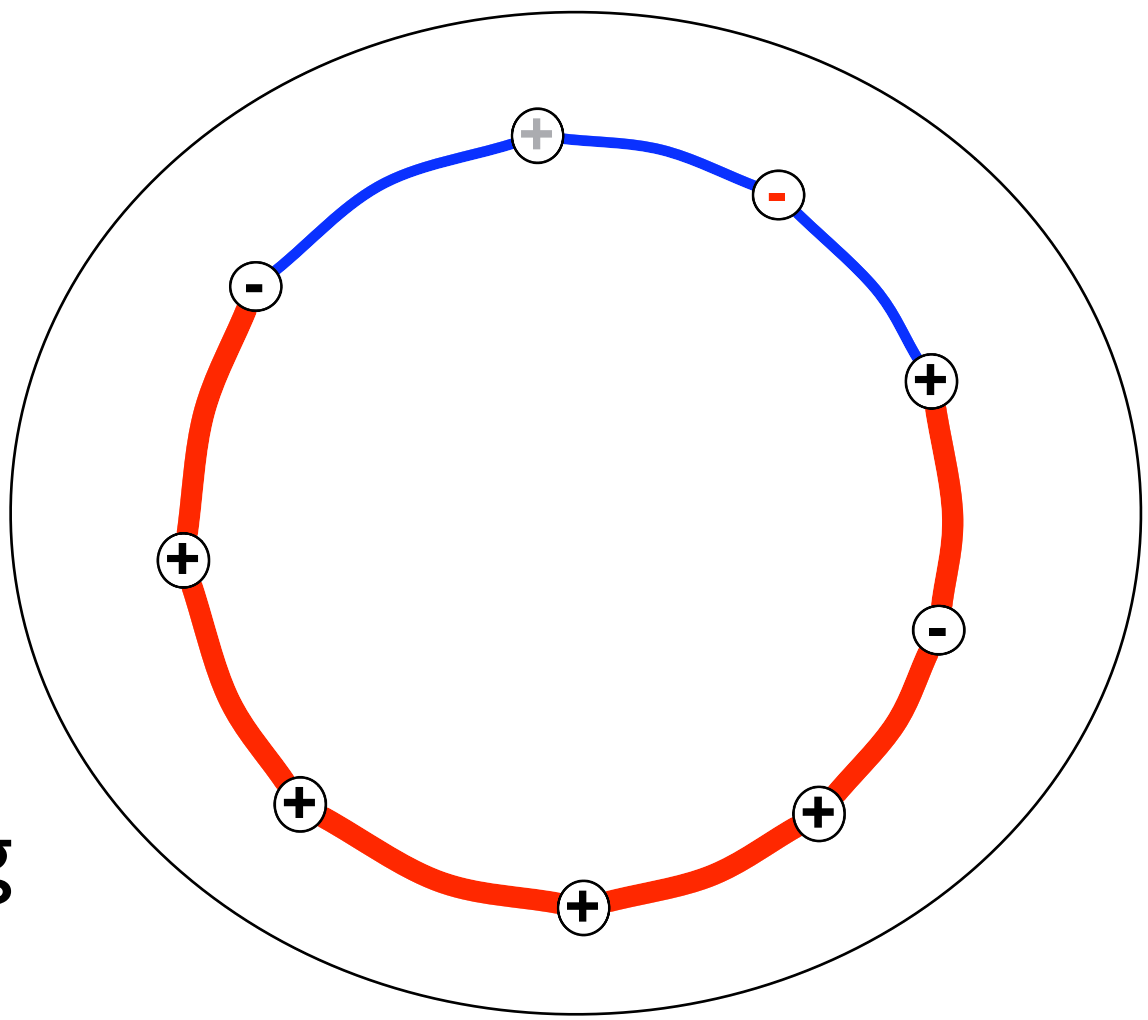


Fixing the Separator

eliminate residual paths from + to - on separator

- make capacity of separator edges infinite
- handle nodes one by one:

- reduce capacity of incident edges back to original
- push + excess to neighbor
- push - excess from neighbor using max-flow in residual graph

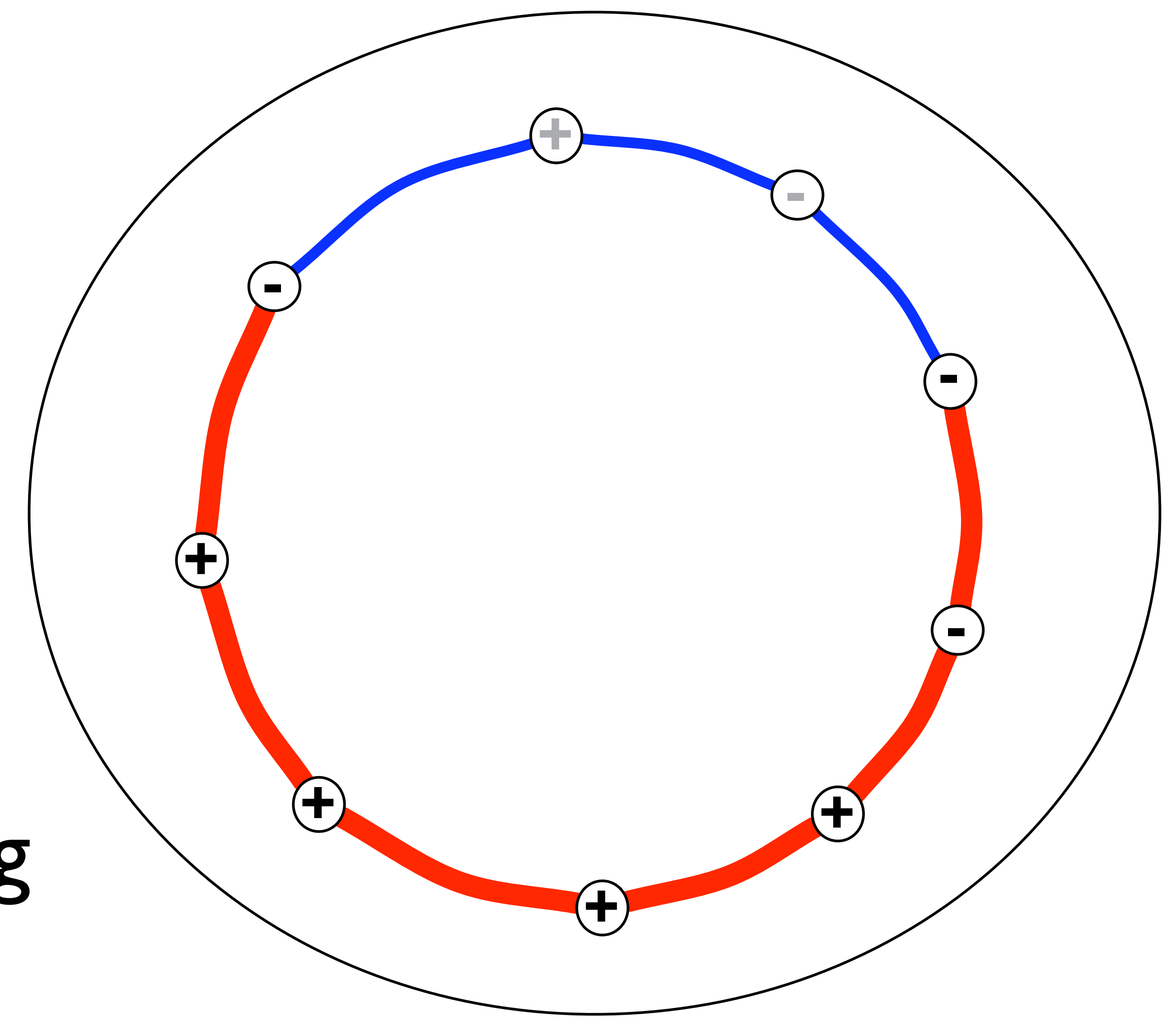


Fixing the Separator

eliminate residual paths from + to - on separator

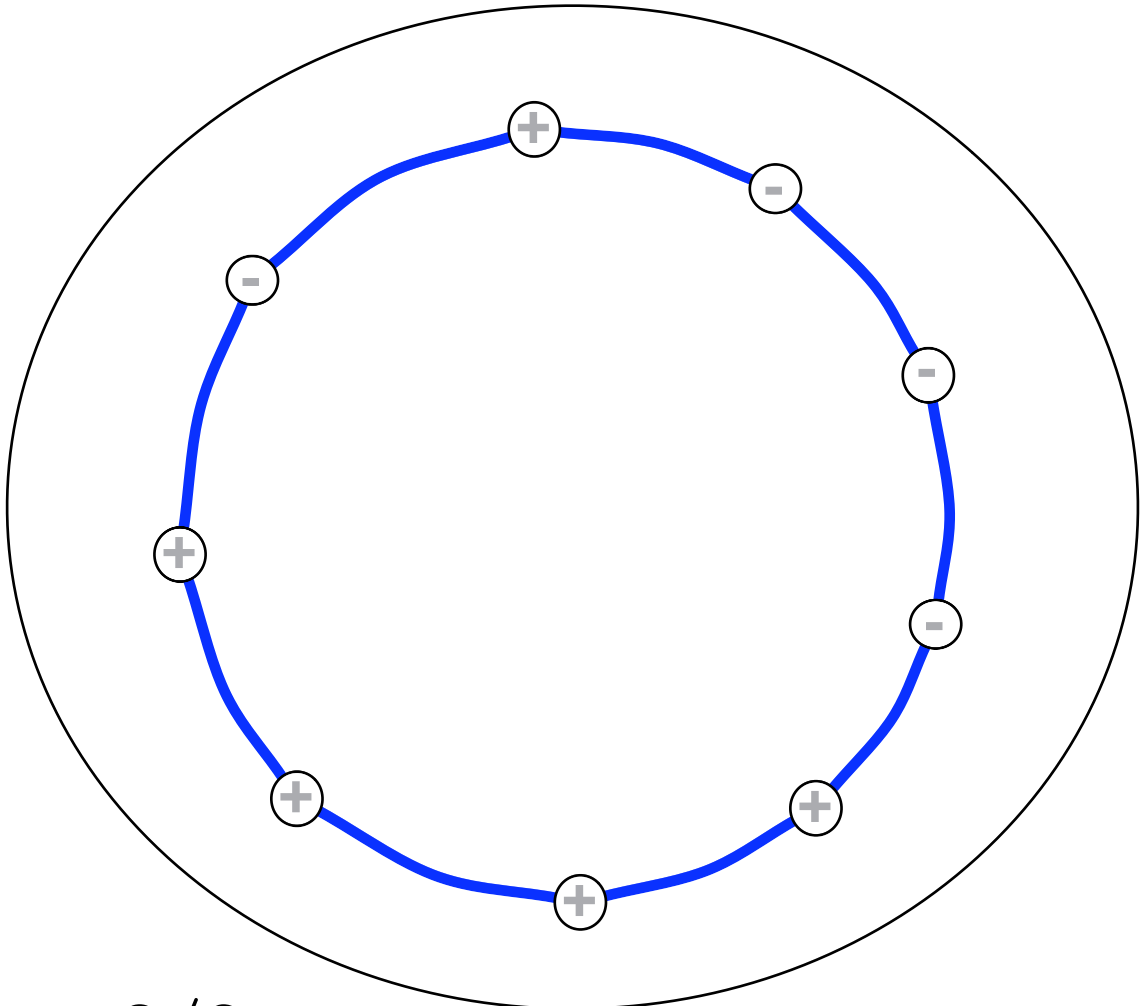
- make capacity of separator edges infinite
- handle nodes one by one:

- reduce capacity of incident edges back to original
- push + excess to neighbor
- push - excess from neighbor using max-flow in residual graph



Fixing the Separator

eliminate residual paths from + to - on separator



- make capacity of separator edges infinite
- handle nodes one by one:

- reduce capacity of incident edges back to original
- push + excess to neighbor
- push - excess from neighbor

running time: $\underbrace{O(\sqrt{n})}_{\text{separator nodes}} \cdot \underbrace{O(n)}_{\text{time for max-flow between neighbors [Hassin + Henzinger et al.]}} = O(n^{3/2})$

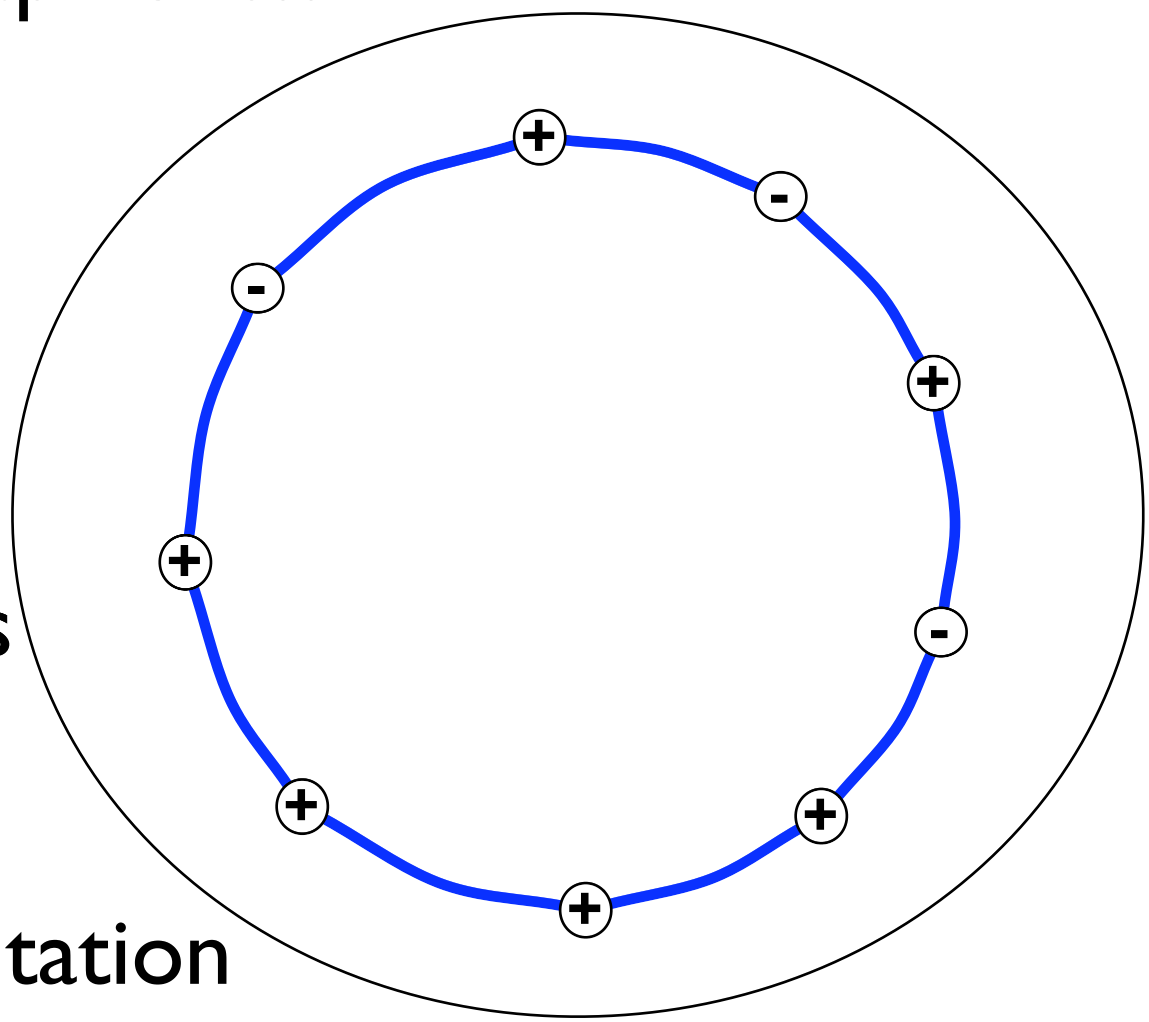
Outline

- a few tools and definitions
- high-level description of recursive algorithm
- main ingredients for near-linear time

Near Linear Time

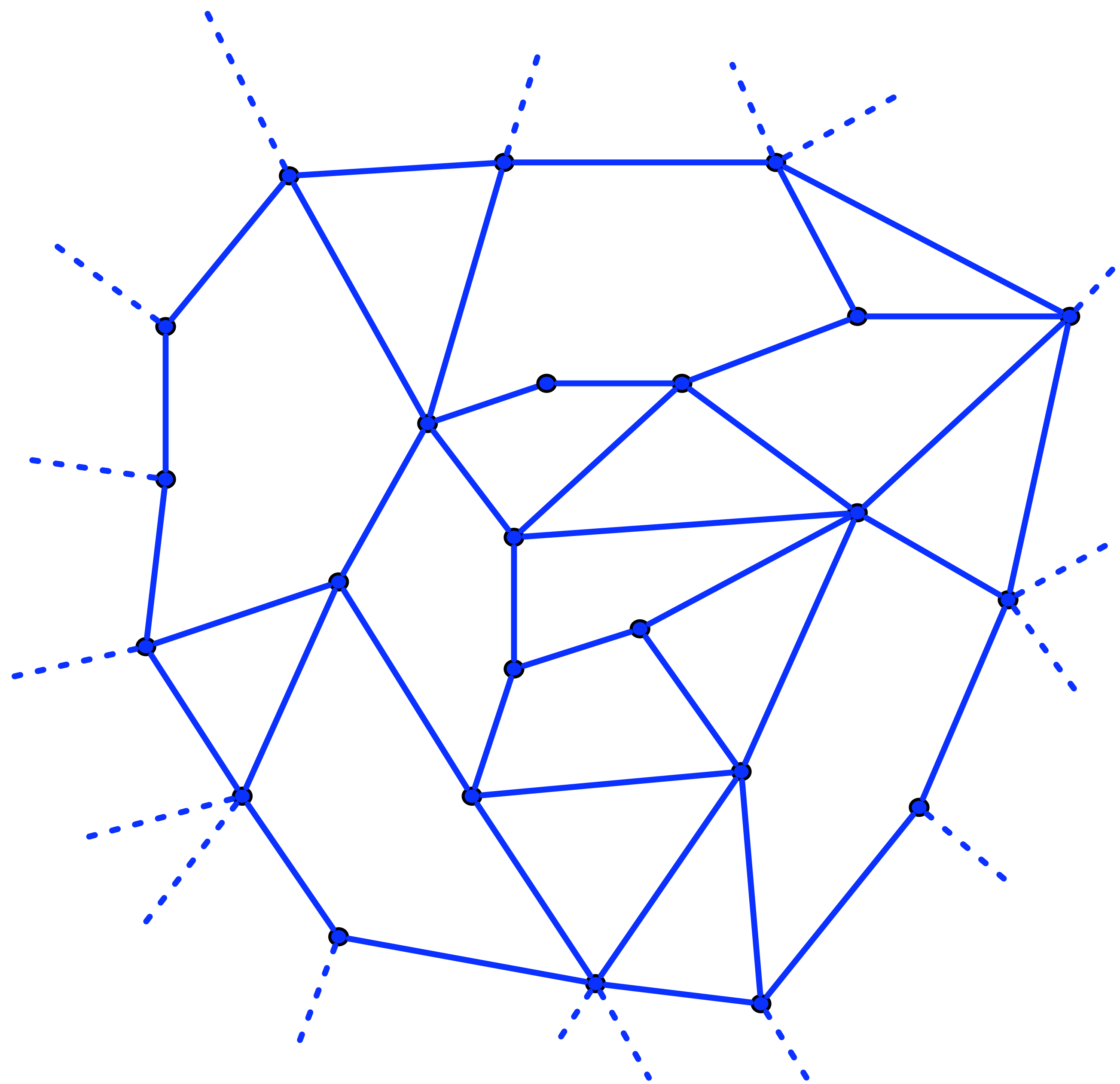
bottleneck is fixing step which consists of $O(\sqrt{n})$ max-flow computations in residual graph between neighbor nodes on a simple cycle

can represent the flow compactly:
flow is in graph with $O(n)$ edges
representation has size $O(\sqrt{n})$
maintain flow only on separator edges
flow elsewhere represented implicitly

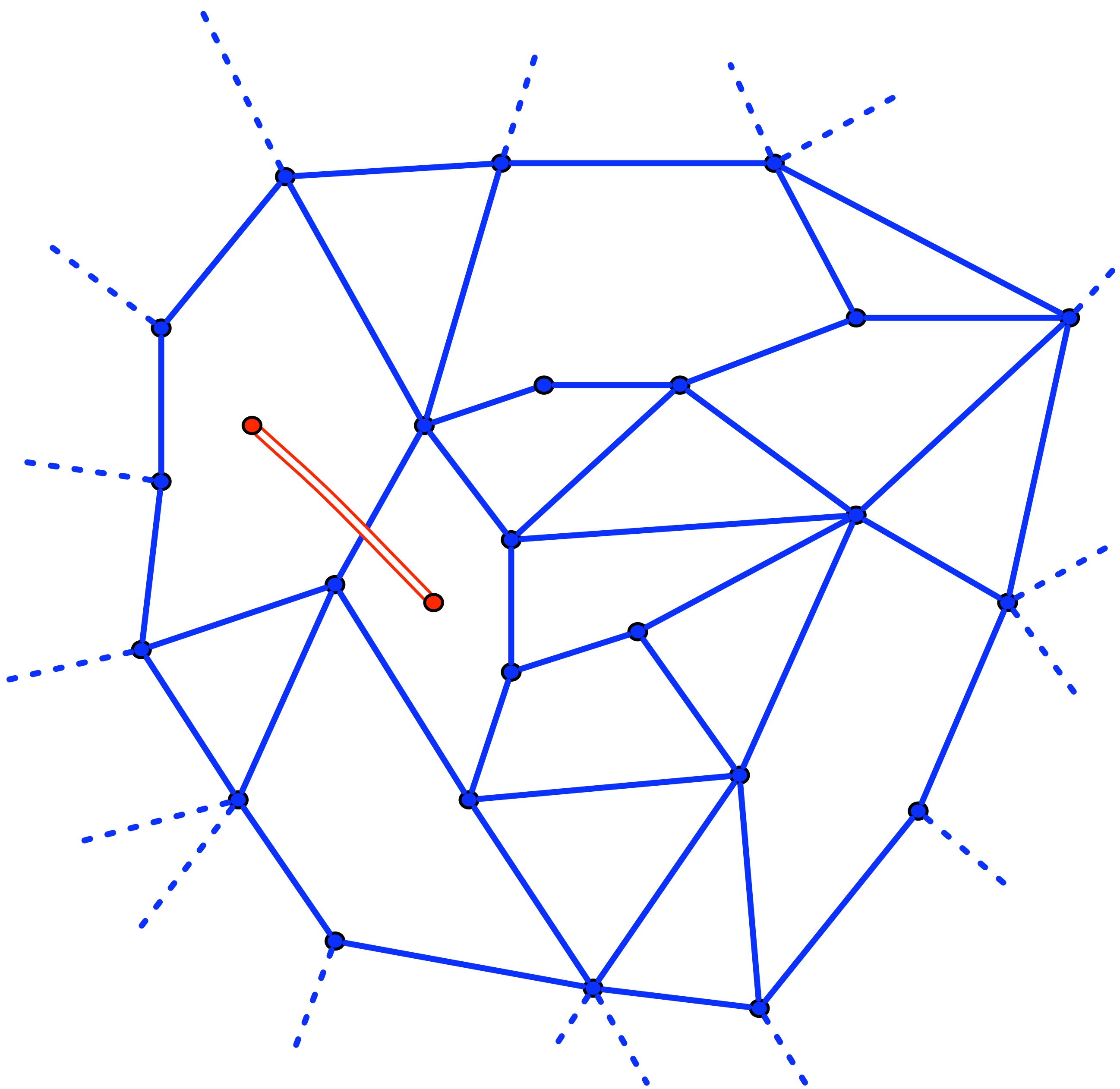


$\Rightarrow$ can perform each max-flow computation in $O(\sqrt{n}\log^2 n)$ instead of $O(n)$

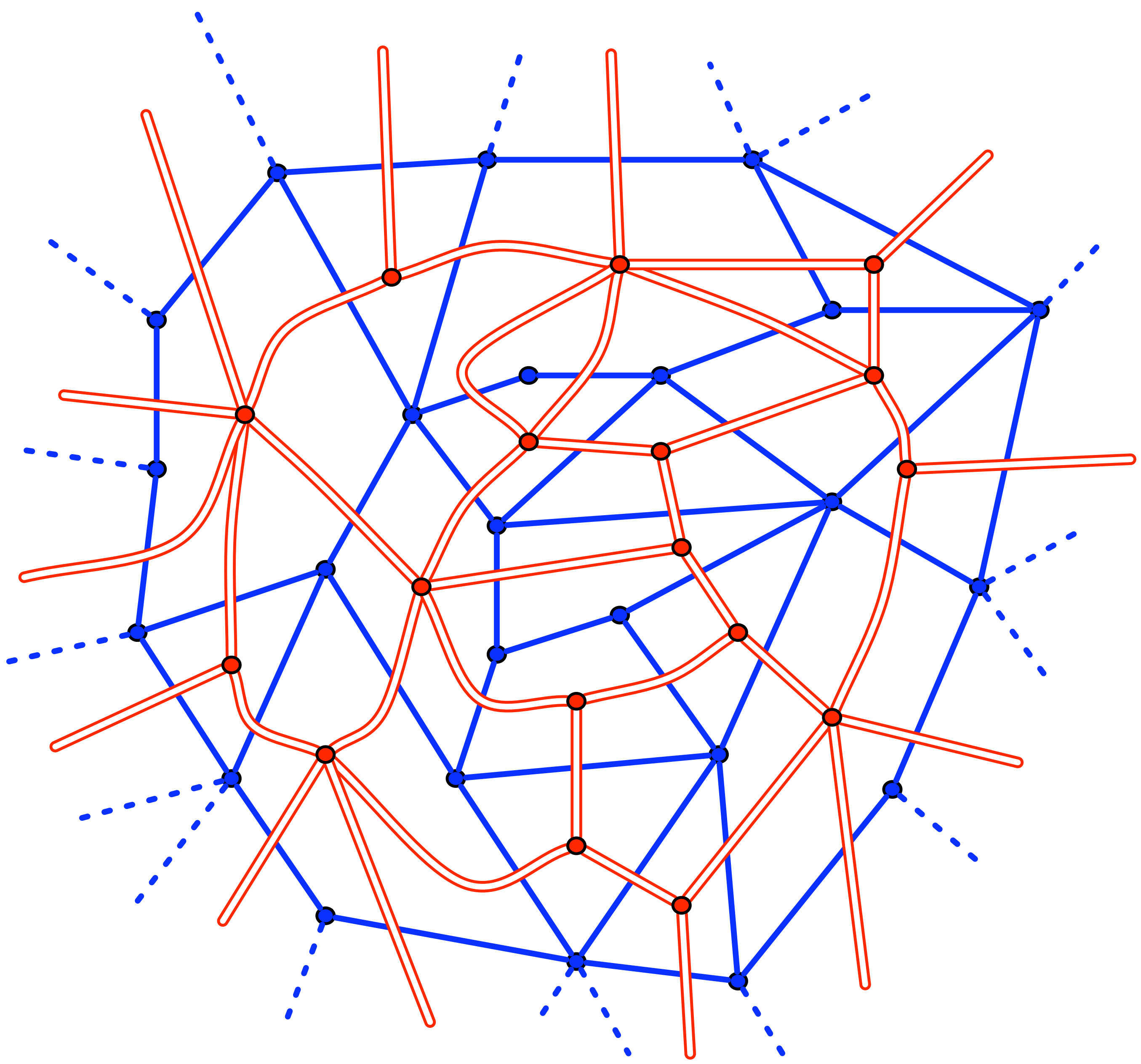
Planar Duality



Planar Duality

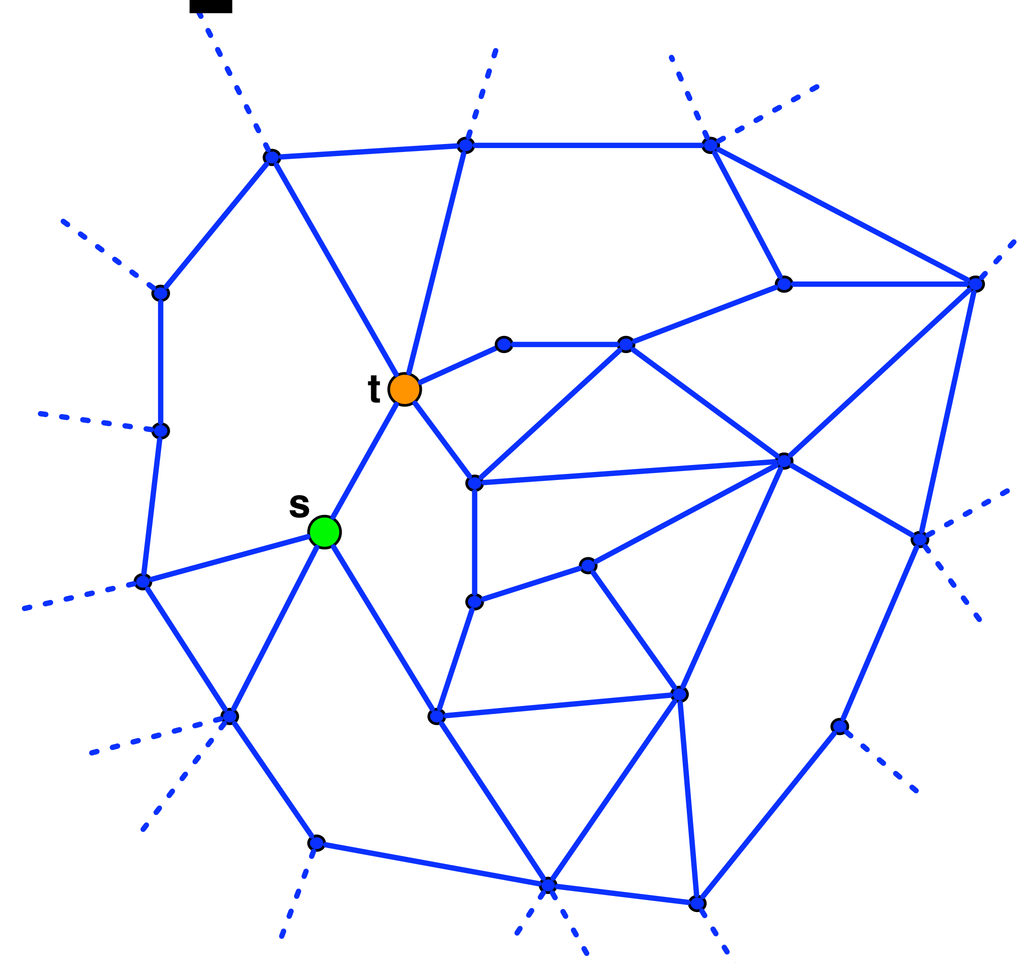


Planar Duality



Max-Flow between Neighbors [Hassin 1981]

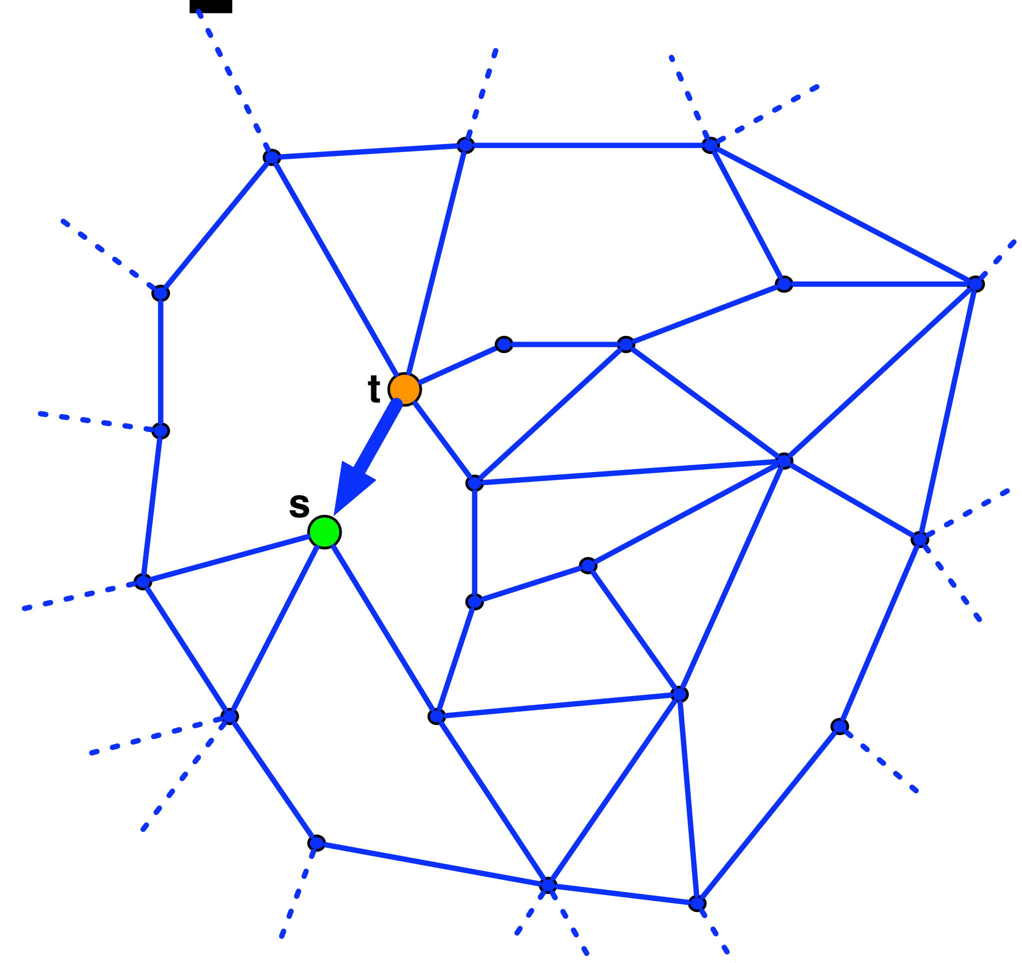
to compute max flow from s to t:



Max-Flow between Neighbors [Hassin 1981]

to compute max flow from s to t:

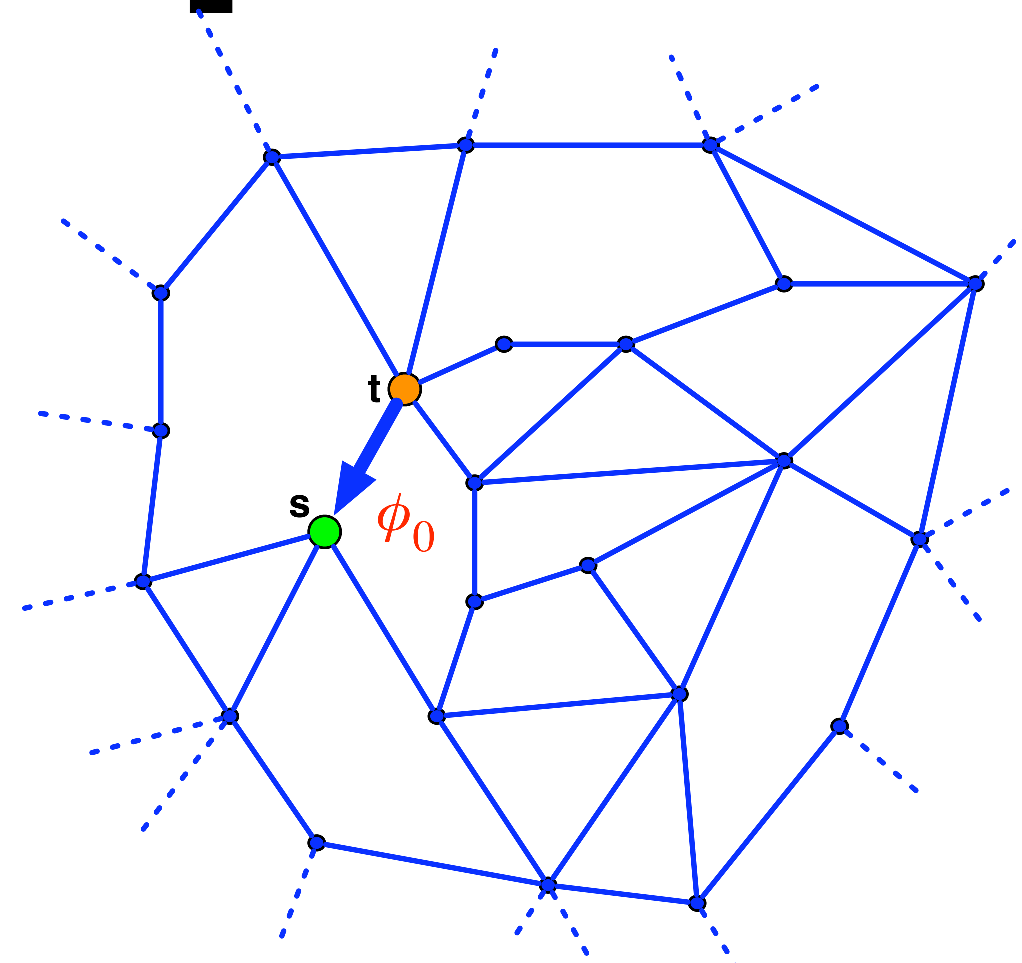
- make capacity of arc ts infinite



Max-Flow between Neighbors [Hassin 1981]

to compute max flow from s to t:

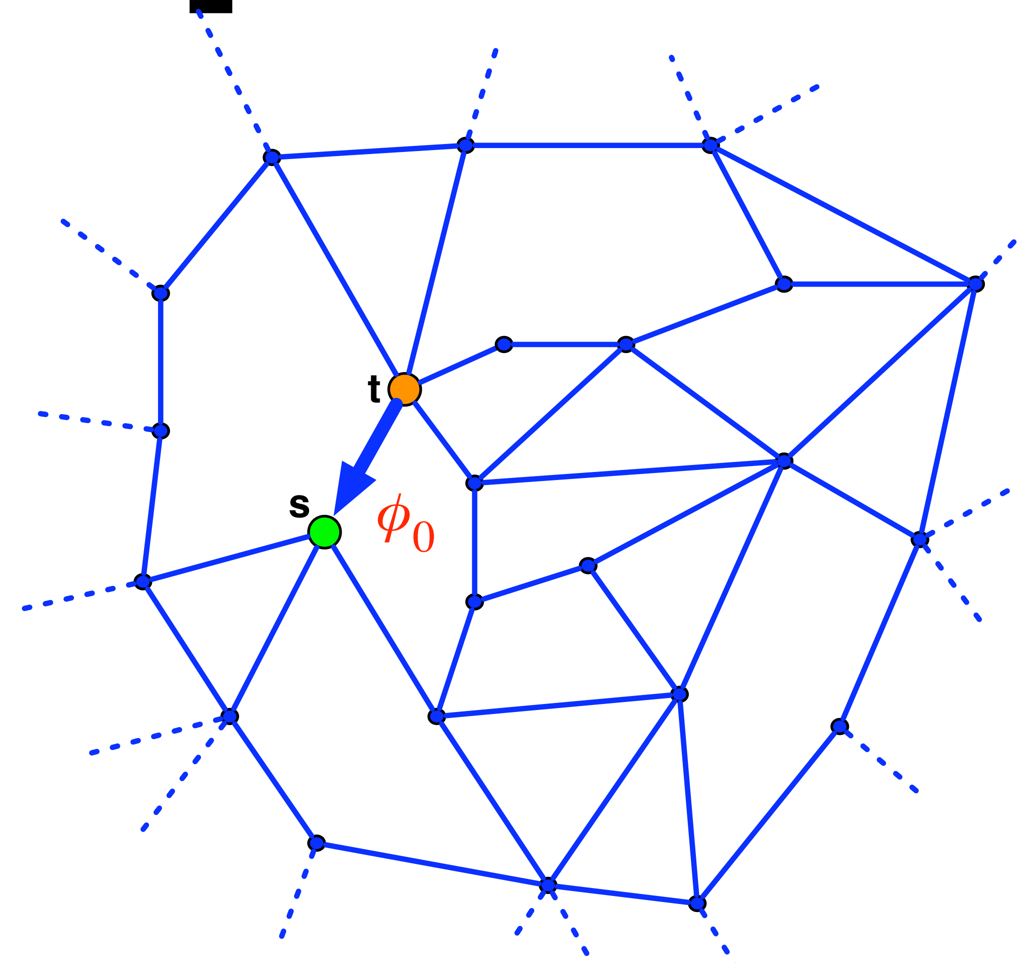
- make capacity of arc ts infinite
- ϕ_0 = the face to the left of arc ts



Max-Flow between Neighbors [Hassin 1981]

to compute max flow from s to t:

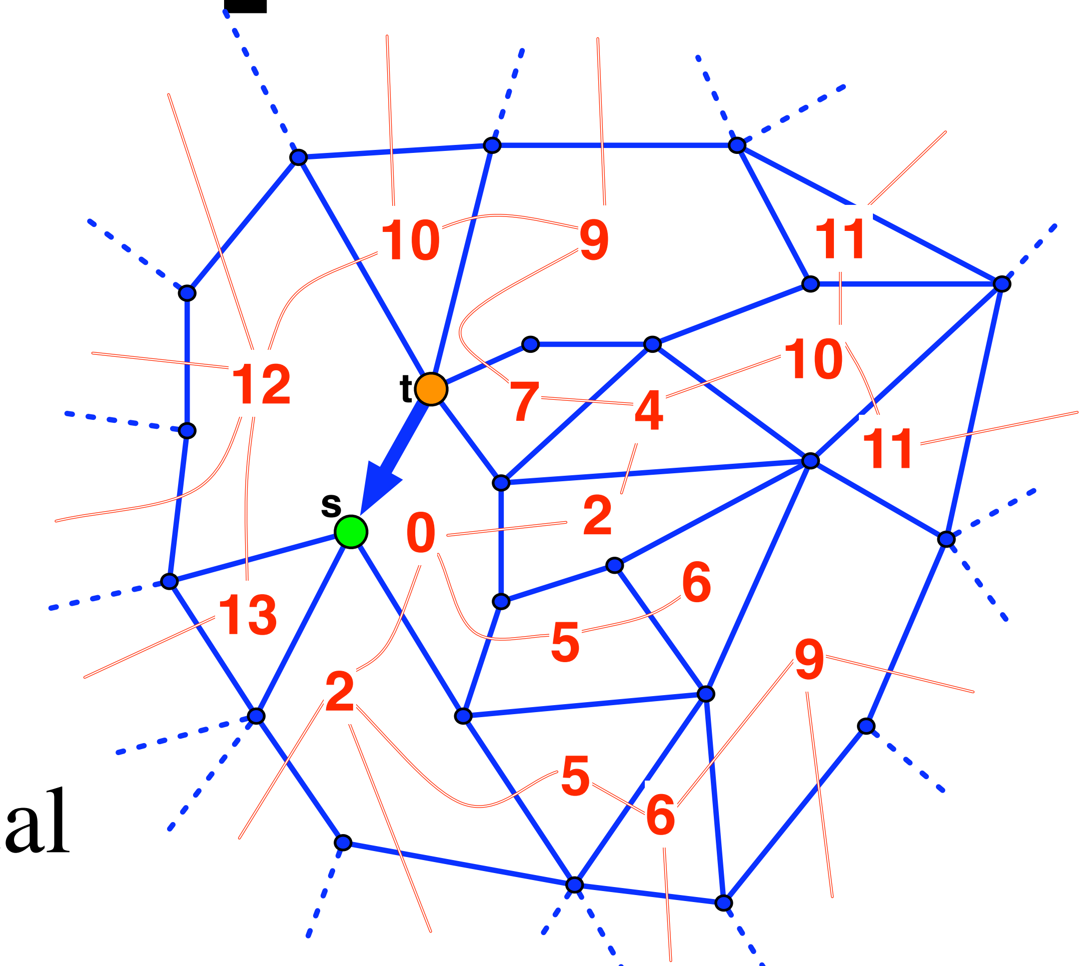
- make capacity of arc ts infinite
- ϕ_0 = the face to the left of arc ts
- consider capacity of an arc in the primal as its length in the dual



Max-Flow between Neighbors [Hassin 1981]

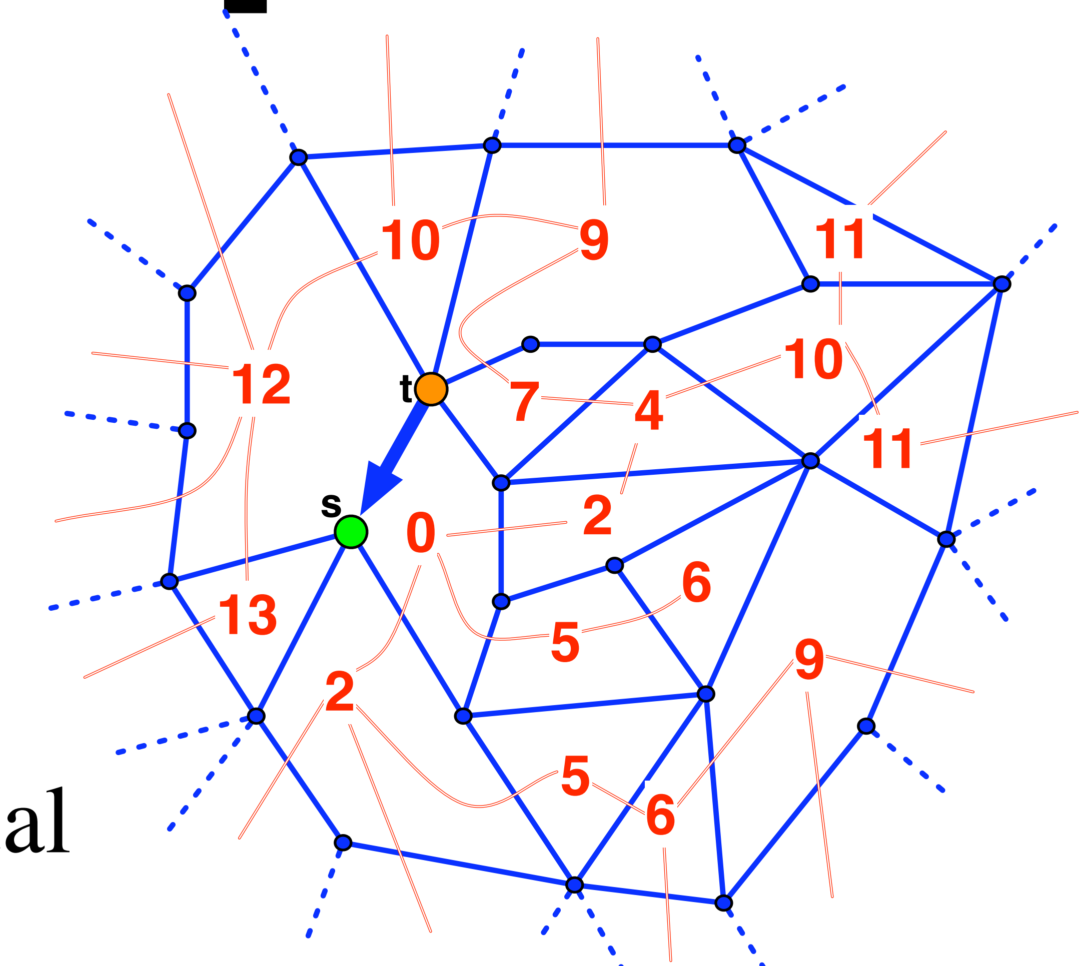
to compute max flow from s to t:

- make capacity of arc ts infinite
- ϕ_0 = the face to the left of arc ts
- consider capacity of an arc in the primal as its length in the dual
- compute:
  $d(\phi)$ = distance of ϕ from ϕ_0 in dual



Max-Flow between Neighbors [Hassin 1981]

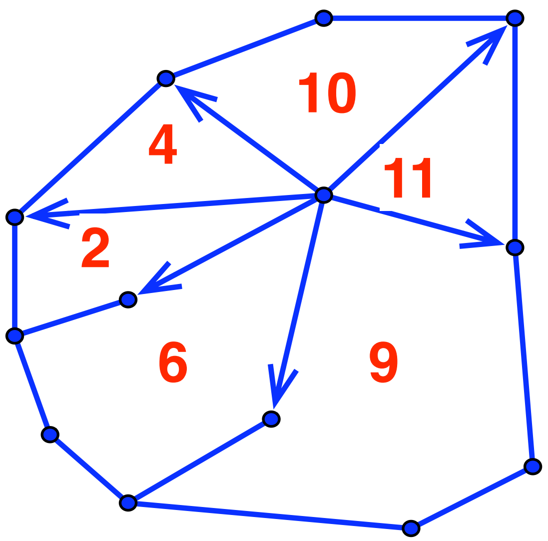
to compute max flow from s to t:



- make capacity of arc ts infinite
- ϕ_0 = the face to the left of arc ts
- consider capacity of an arc in the primal as its length in the dual
- compute:
  $d(\phi)$ = distance of ϕ from ϕ_0 in dual
- define flow on arc a by:
  $\sigma(a) = d(\text{face right of } a) - d(\text{face left of } a)$
  σ is a feasible circulation that maximizes the flow on arc ts

σ is a feasible circulation

σ is a feasible circulation



conservation:

$\sigma(a) = d$(face right of a) - d(face left of a)

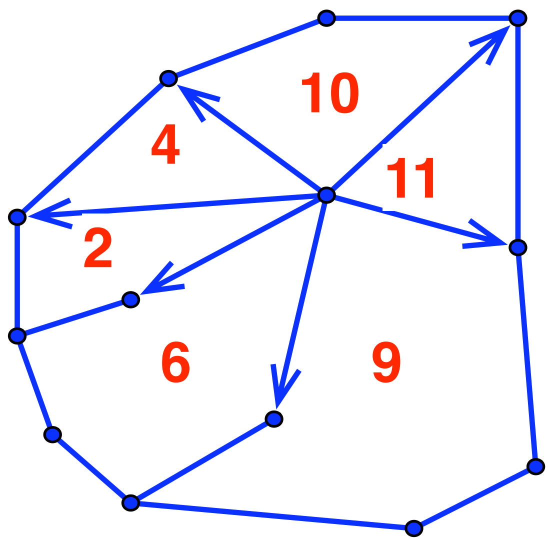
flows on arcs outgoing from a node cancel to zero

σ is a feasible circulation

conservation:

$\sigma(a) = d(\text{face right of } a) - d(\text{face left of } a)$

flows on arcs outgoing from a node cancel to zero



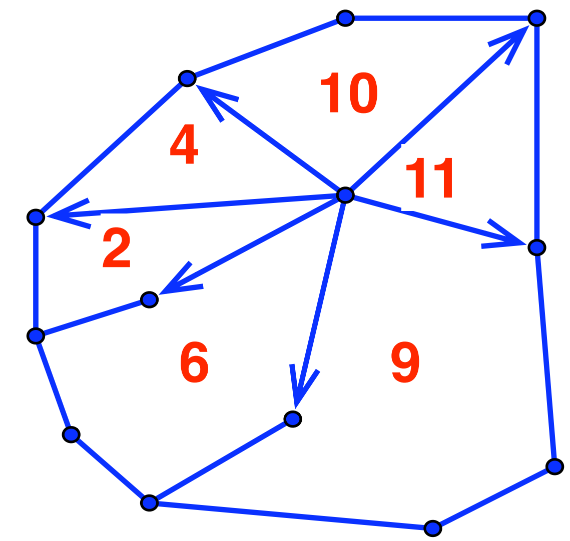
feasibility guaranteed by shortest paths inequality:

σ is a feasible circulation

conservation:

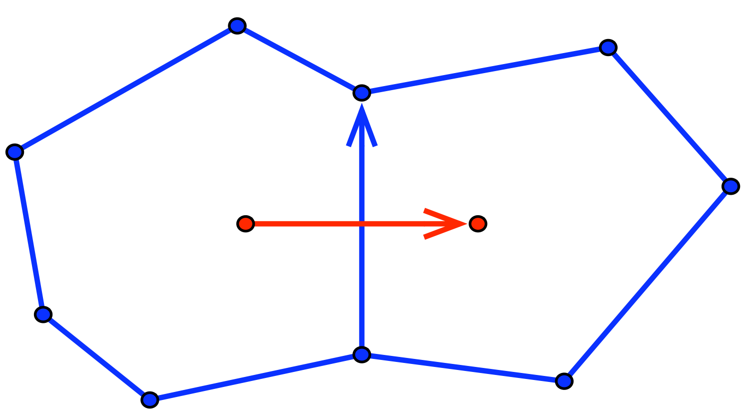
$\sigma(a) = d(\text{face right of } a) - d(\text{face left of } a)$

flows on arcs outgoing from a node cancel to zero



feasibility guaranteed by shortest paths inequality:

$d(\text{head of dual of } a) \leq d(\text{tail of dual of } a) + \text{length}(\text{dual of } a)$

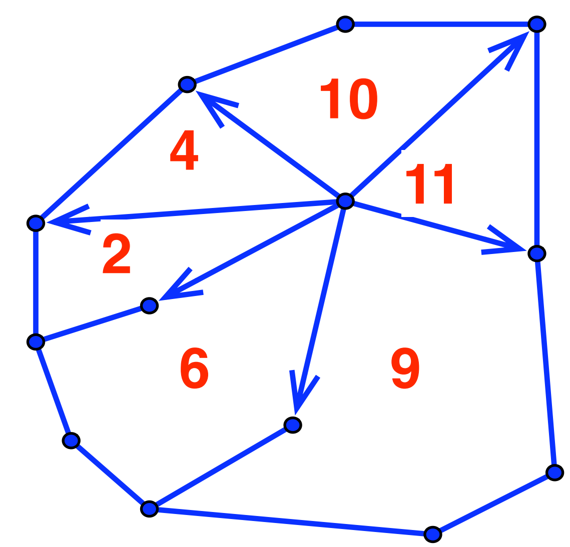


σ is a feasible circulation

conservation:

$\sigma(a) = d(\text{face right of } a) - d(\text{face left of } a)$

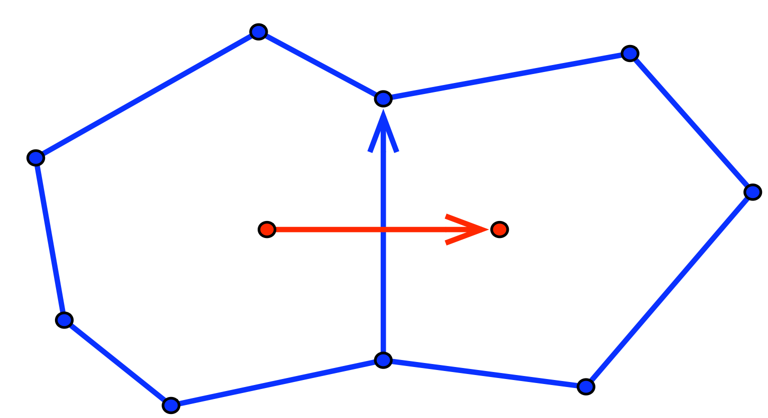
flows on arcs outgoing from a node cancel to zero



feasibility guaranteed by shortest paths inequality:

$d(\text{head of dual of } a) \leq d(\text{tail of dual of } a) + \text{length}(\text{dual of } a)$

$d(\text{head of dual of } a) - d(\text{tail of dual of } a) \leq \text{capacity of } a$

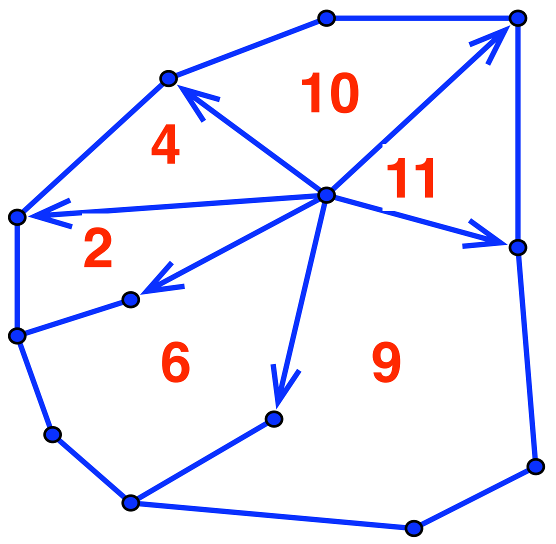


σ is a feasible circulation

conservation:

$\sigma(a) = d$(face right of a) - d(face left of a)

flows on arcs outgoing from a node cancel to zero



feasibility guaranteed by shortest paths inequality:

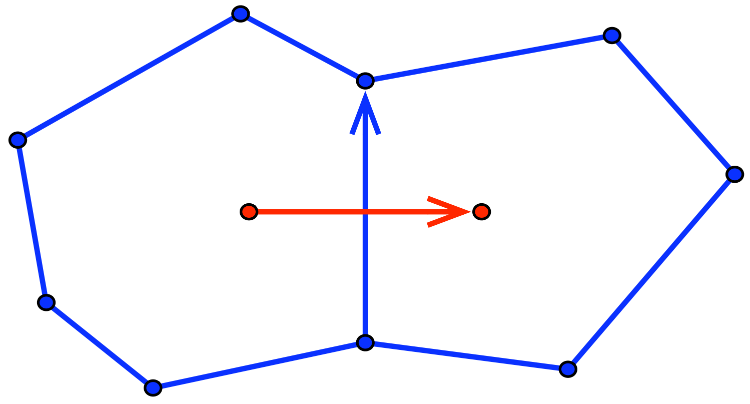
d(head of dual of a) $\leq$ d(tail of dual of a) + length(dual of a)

d(head of dual of a) - d(tail of dual of a) $\leq$ capacity of a

$\sigma(a) = d$(face right of a) - d(face left of a)

$= d$(head of dual of a) - d(tail of dual of a)

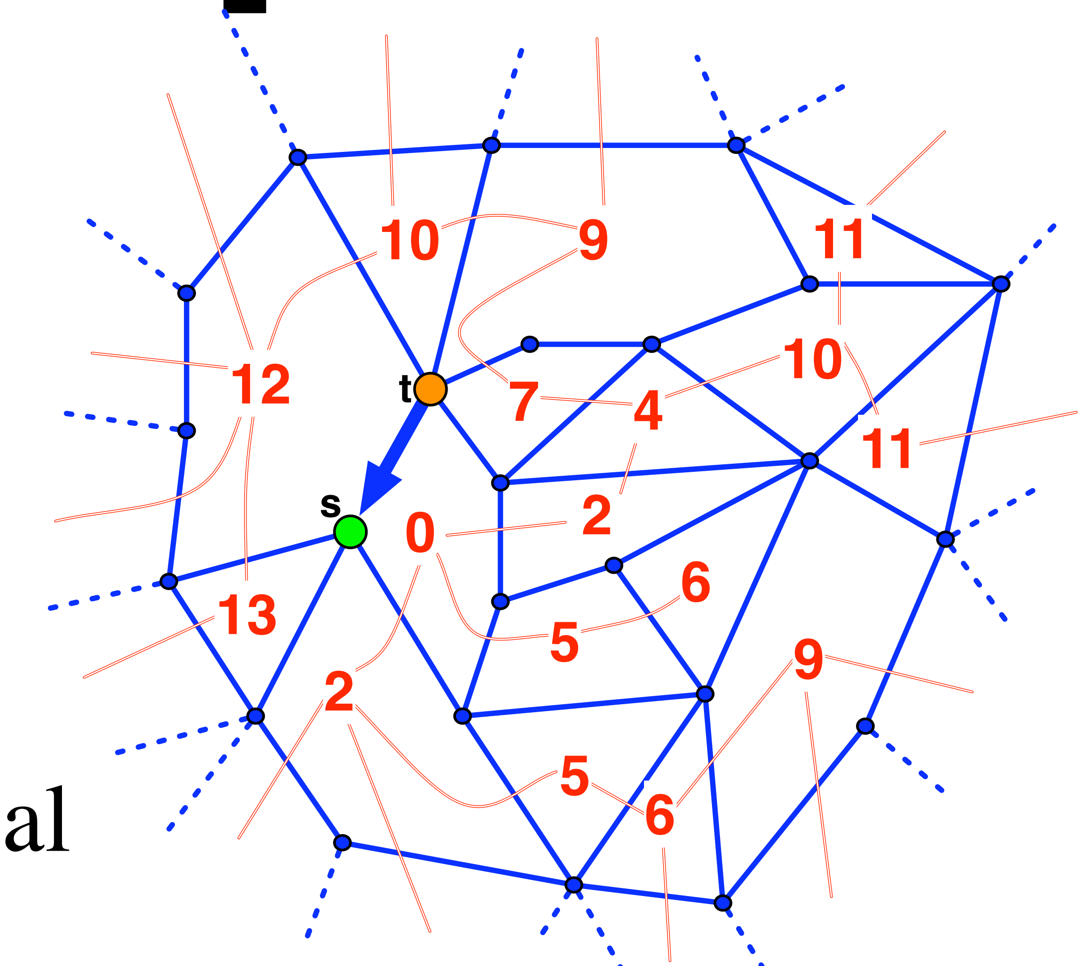
$\leq$ capacity of a



Max-Flow between Neighbors [Hassin 1981]

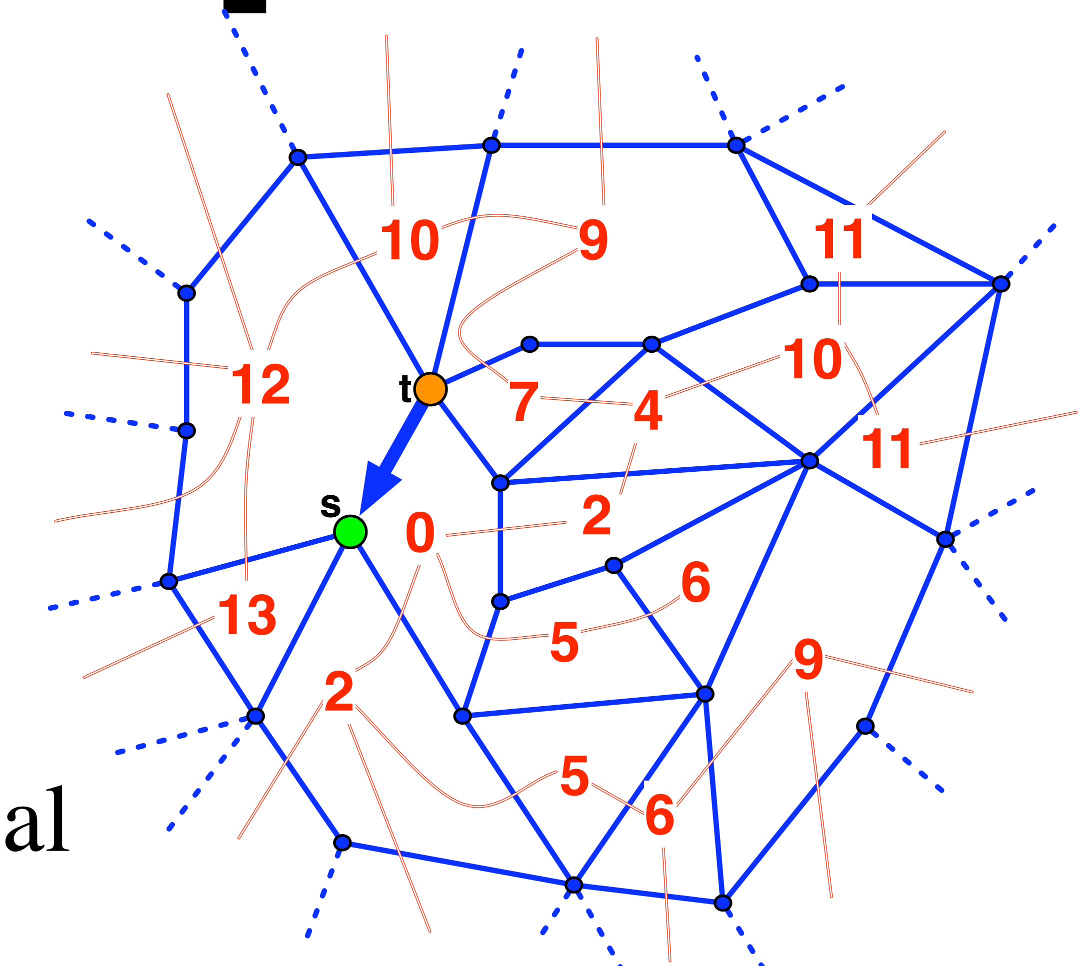
to compute max flow from s to t:

- make capacity of arc ts infinite
- ϕ_0 = the face to the left of arc ts
- consider capacity of an arc in the primal as its length in the dual
- compute:
  $d(\phi)$ = distance of ϕ from ϕ_0 in dual
- define flow on arc a by:
  $\sigma(a) = d$(face right of a) - d(face left of a)
  σ is a feasible circulation that maximizes the flow on arc ts



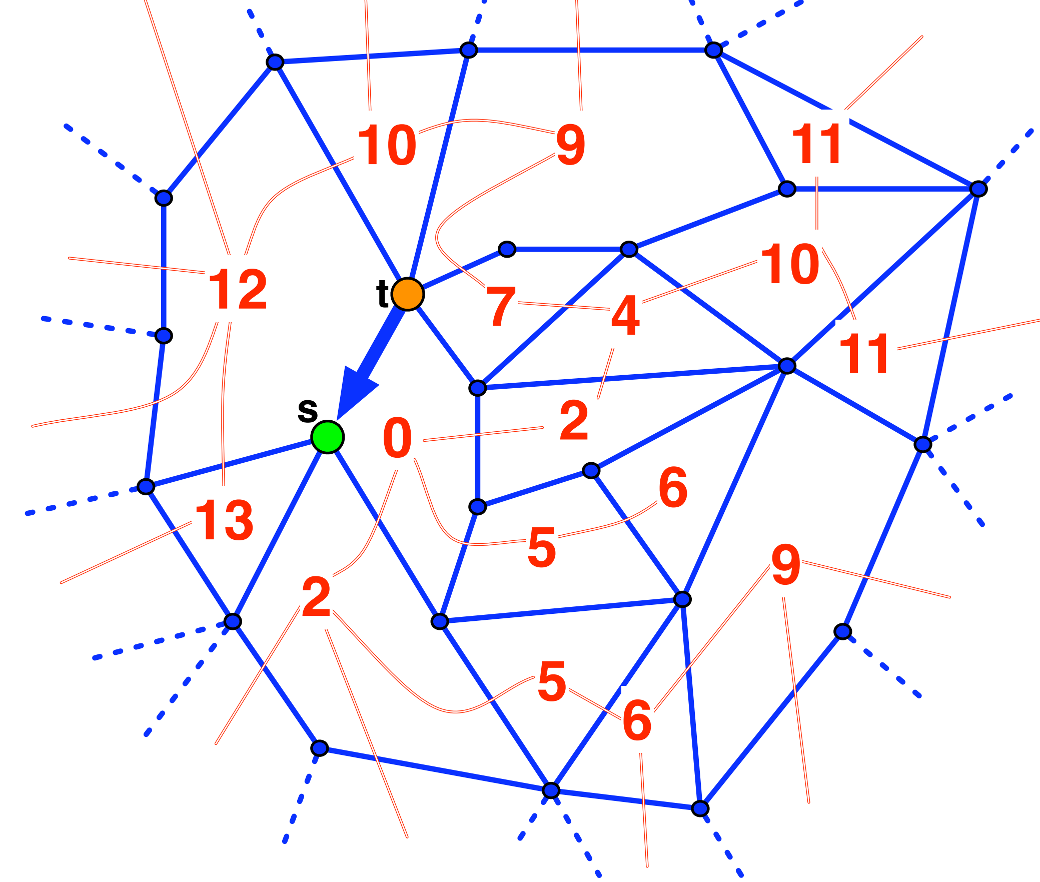
Max-Flow between Neighbors [Hassin 1981]

to compute max flow from s to t:

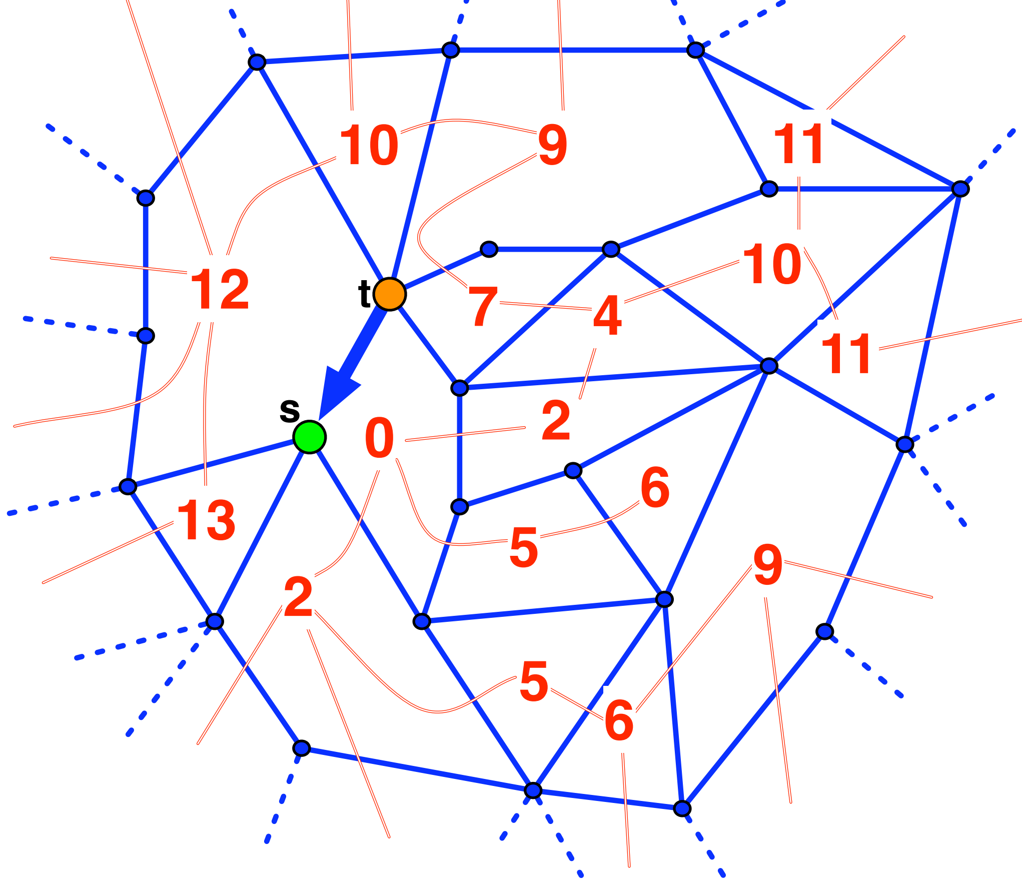


- make capacity of arc ts infinite
- ϕ_0 = the face to the left of arc ts
- consider capacity of an arc in the primal as its length in the dual
- compute:
  $d(\phi)$ = distance of ϕ from ϕ_0 in dual
- define flow on arc a by:
  $\sigma(a) = d(\text{face right of } a)$ - $d(\text{face left of } a)$
  σ is a feasible circulation that maximizes the flow on arc ts
- don't push flow on ts

Flow Representation 1/4

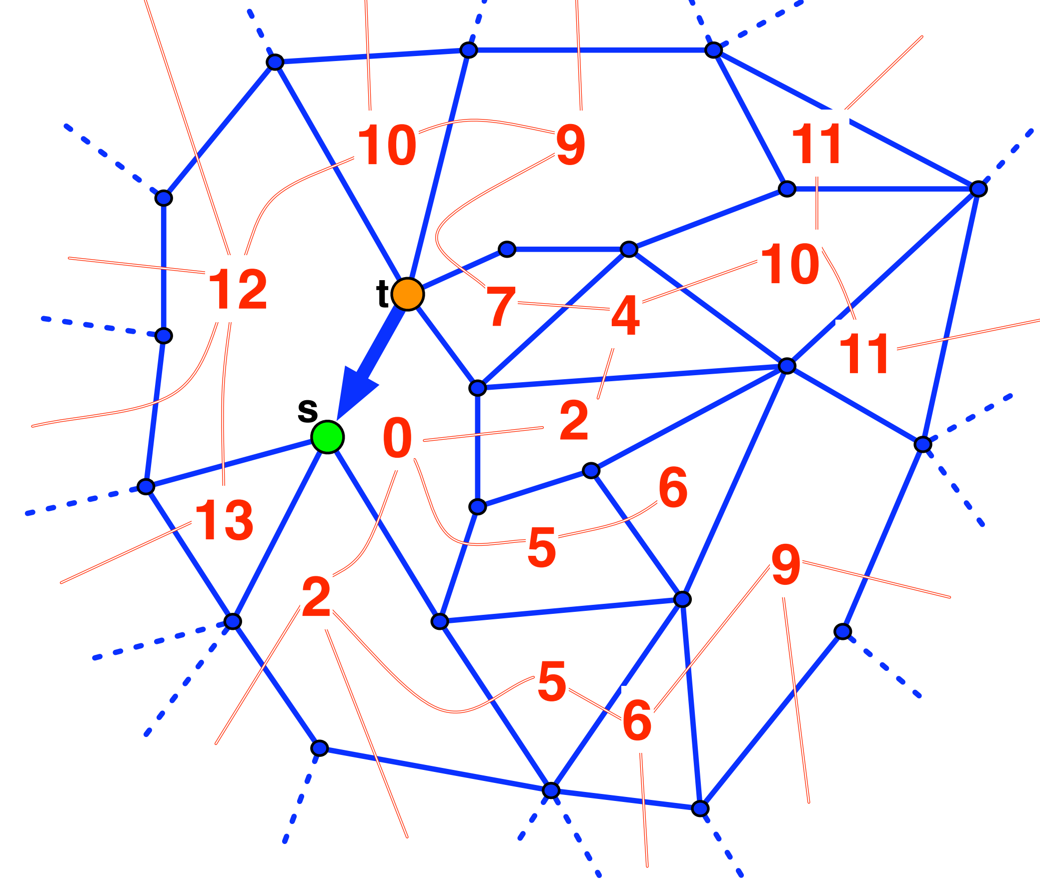


Flow Representation 1/4



flow between endpoints of arc a_i of C can be represented by:

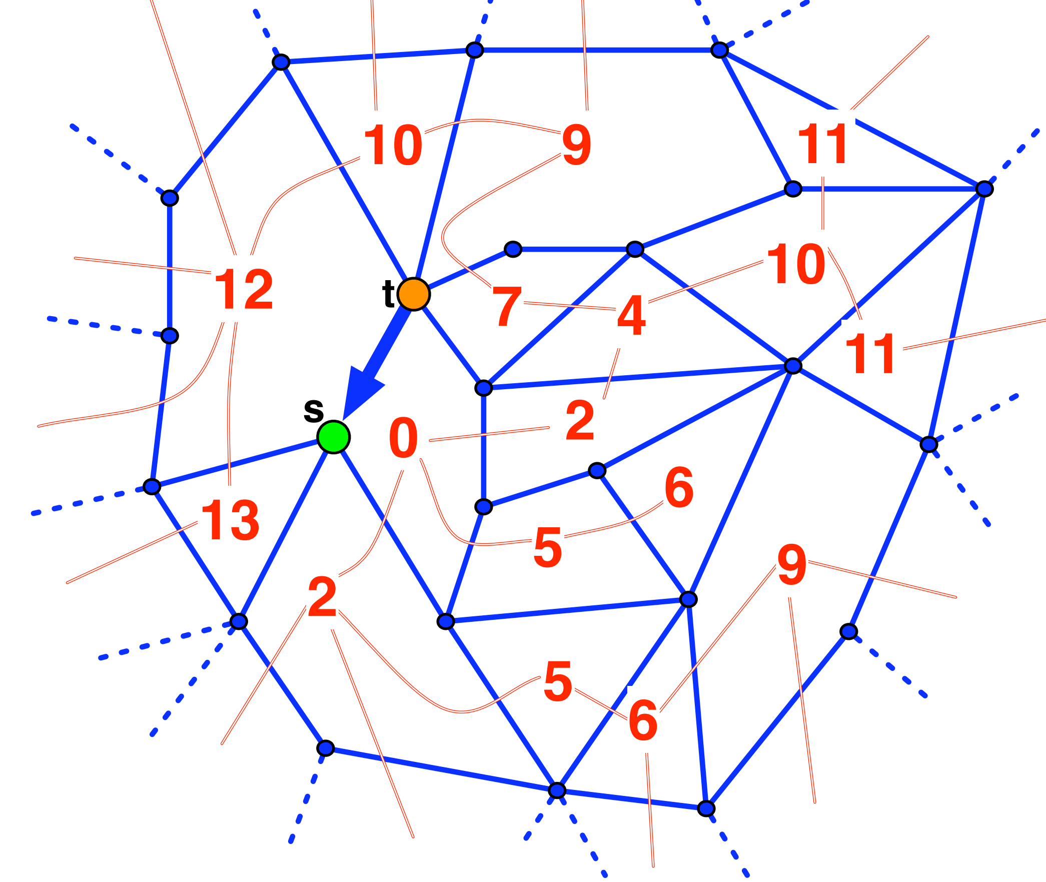
Flow Representation 1/4



flow between endpoints of arc a_i of C can be represented by:

- face labels $d_i(\phi)$ for each face ϕ

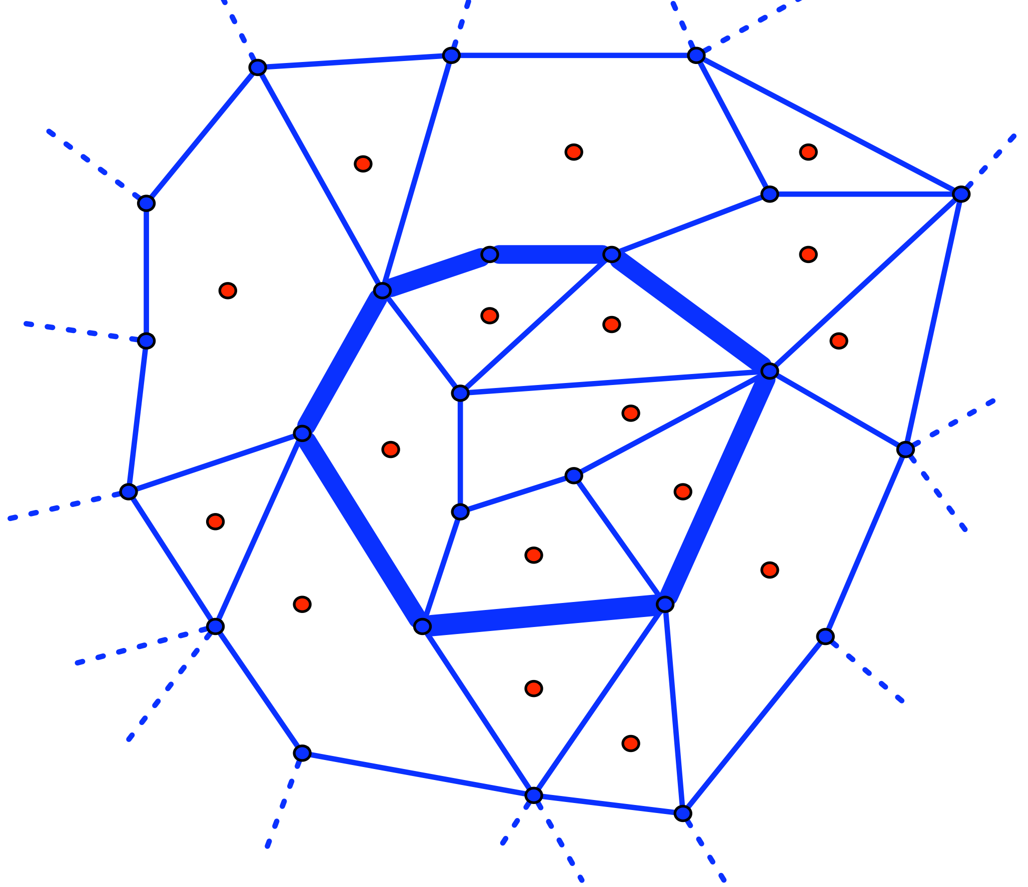
Flow Representation 1/4



flow between endpoints of arc a_i of C can be represented by:

- face labels $d_i(\phi)$ for each face ϕ
- flow on a_i

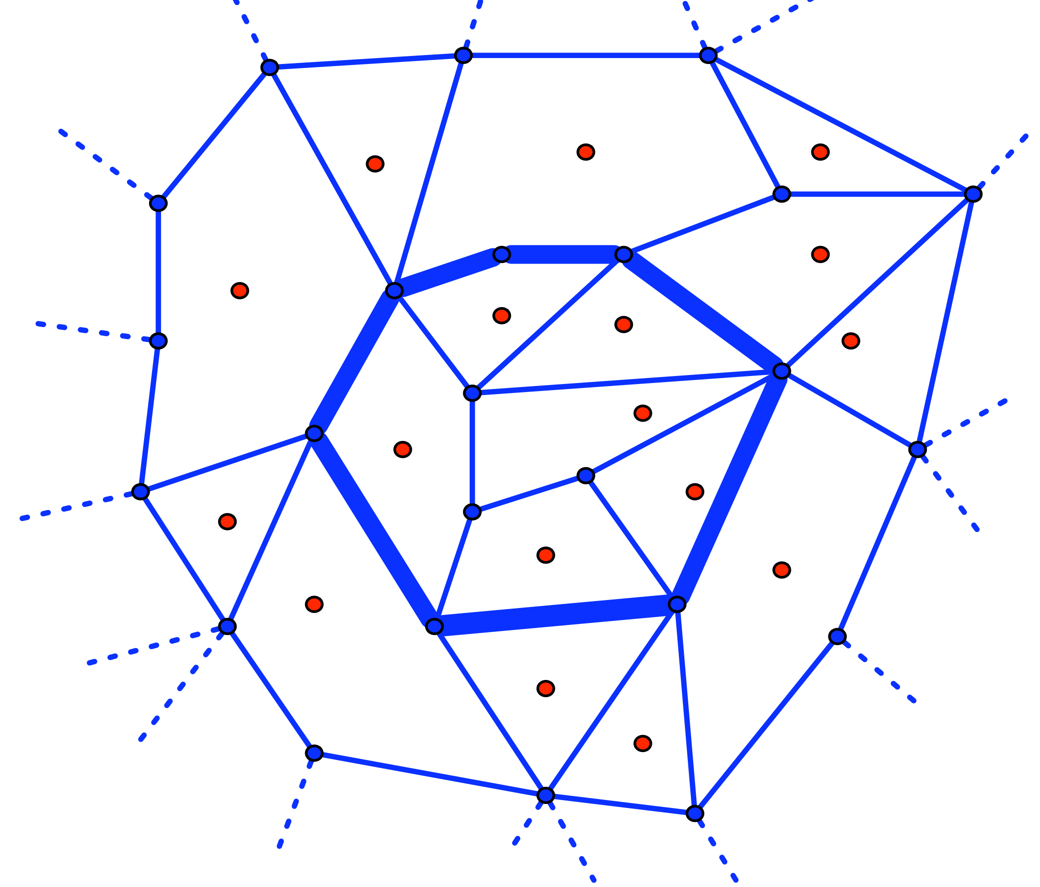
Flow Representation 1/4



flow between endpoints of arc a_i of C can be represented by:

- face labels $d_i(\phi)$ for each face ϕ
- flow on a_i

Flow Representation 1/4

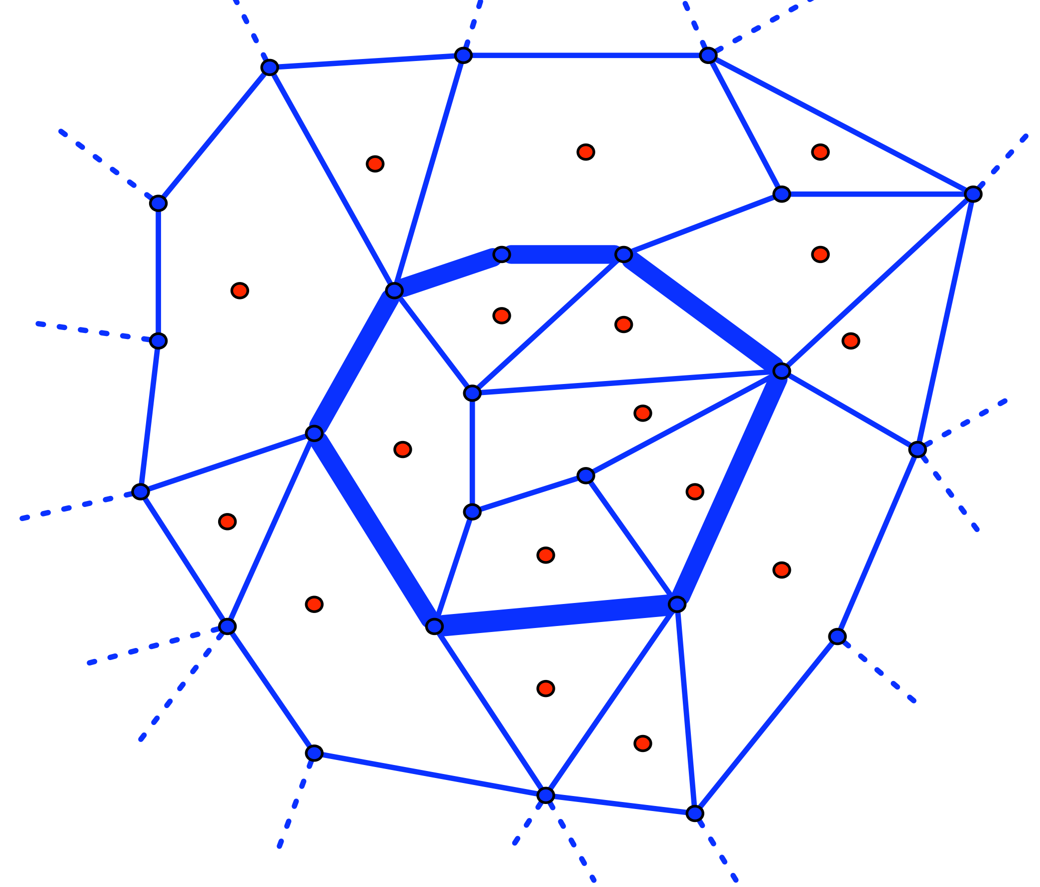


flow between endpoints of arc a_i of C
can be represented by:

- face labels $d_i(\phi)$ for each face ϕ
- flow on a_i

to represent sum of flows for all iterations of fixing step:

Flow Representation 1/4



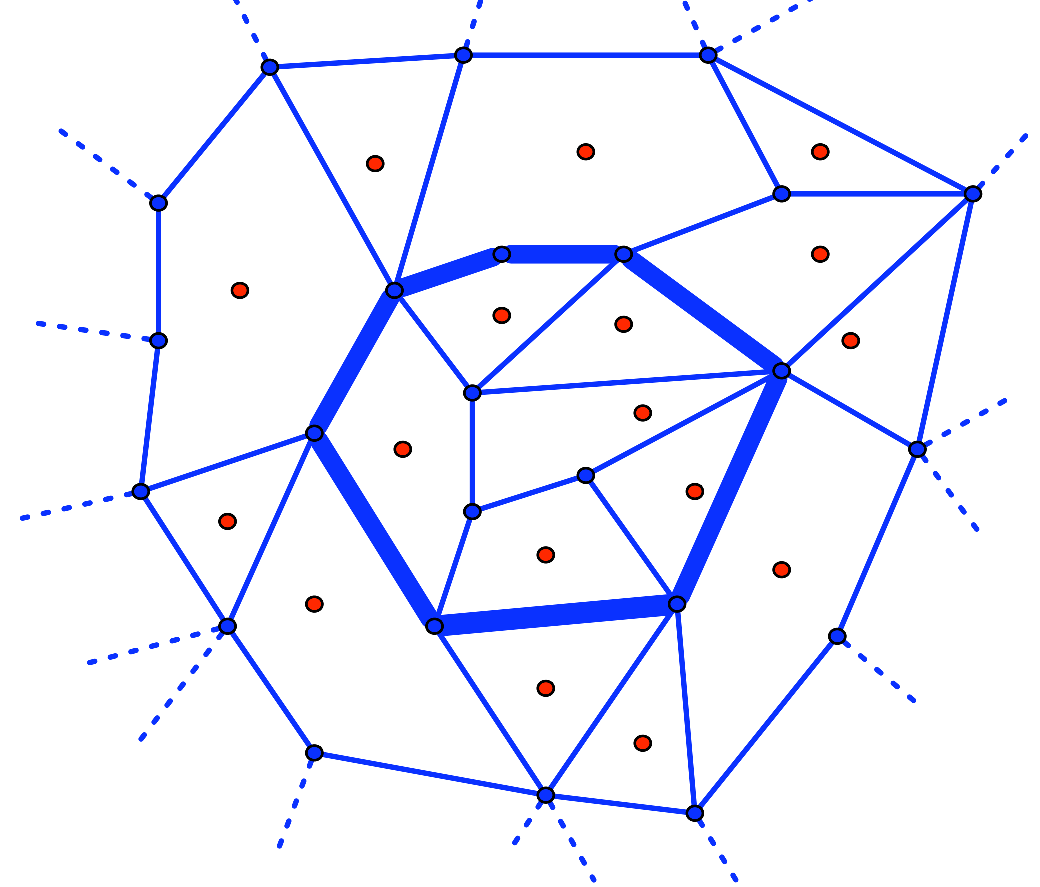
flow between endpoints of arc a_i of C can be represented by:

- face labels $d_i(\phi)$ for each face ϕ
- flow on a_i

to represent sum of flows for all iterations of fixing step:

- accumulate face labels over all iterations (linearity)

Flow Representation 1/4



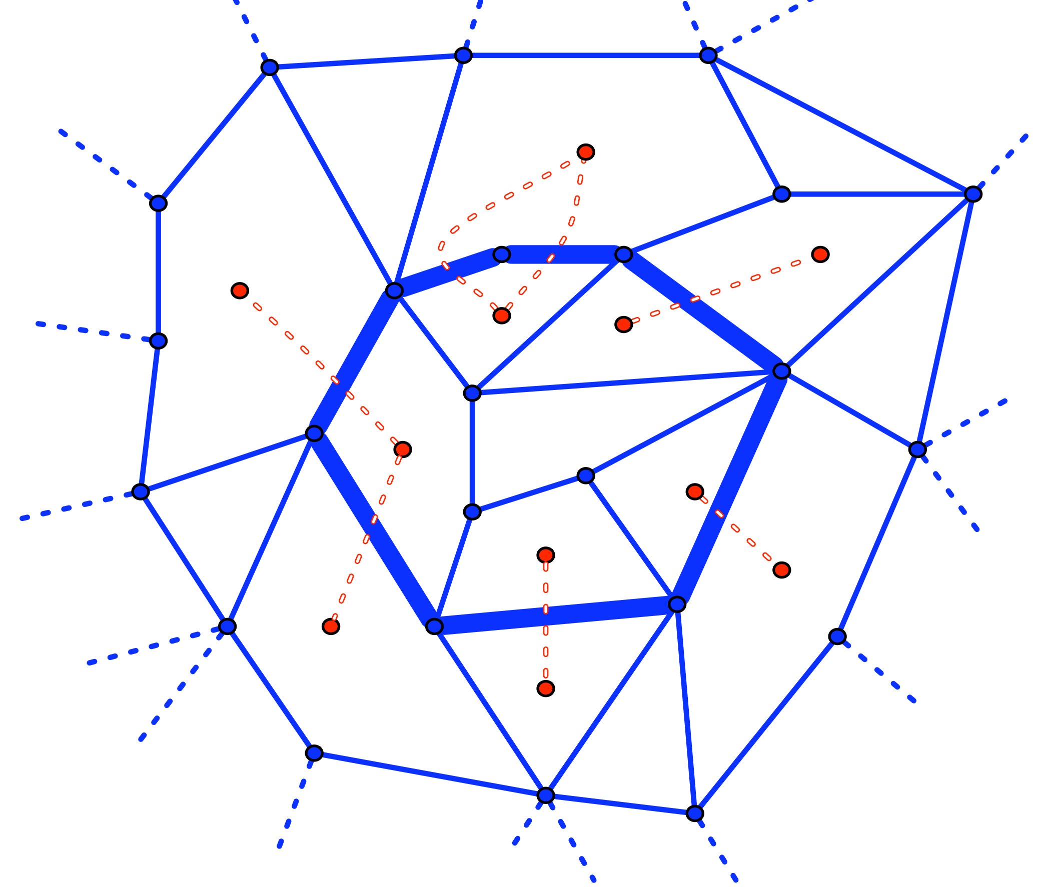
flow between endpoints of arc a_i of C
can be represented by:

- face labels $d_i(\phi)$ for each face ϕ
- flow on a_i

to represent sum of flows for all iterations of fixing step:

- accumulate face labels over all iterations (linearity)
- explicitly store flow on arcs of separator C

Flow Representation 1/4



flow between endpoints of arc a_i of C
can be represented by:

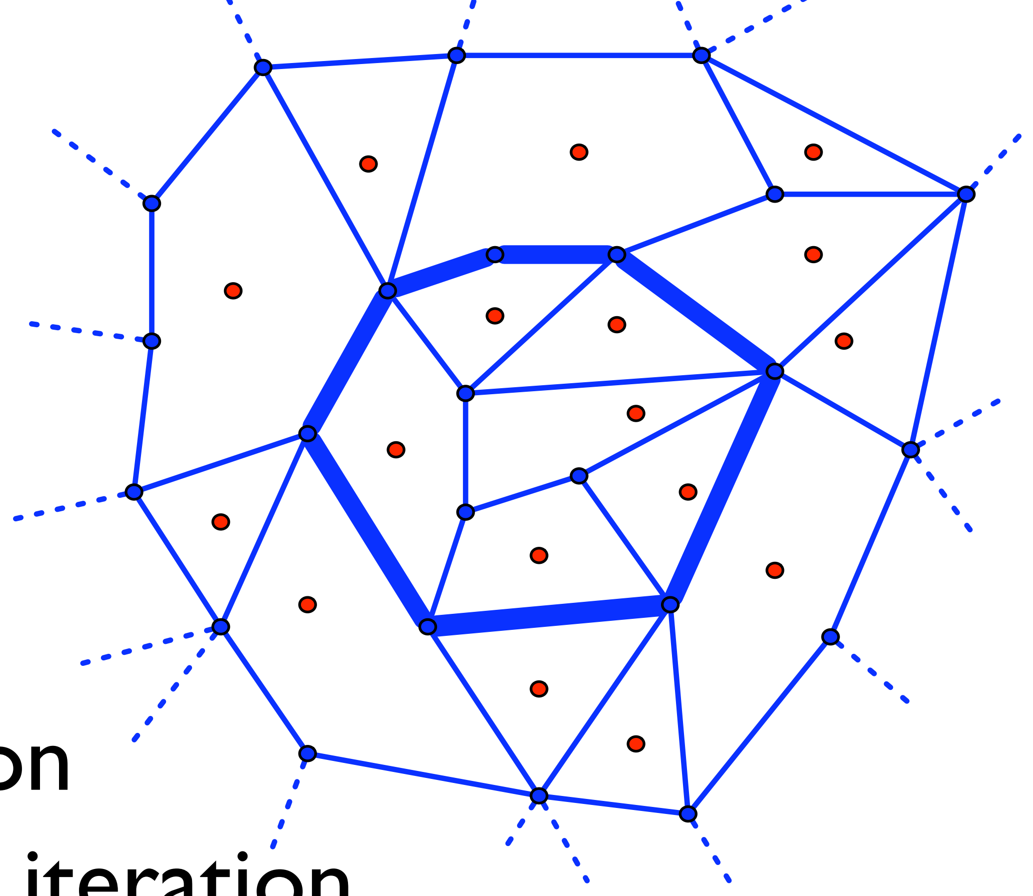
- face labels $d_i(\phi)$ for each face ϕ
- flow on a_i

to represent sum of flows for all iterations of fixing step:

- accumulate face labels over all iterations (linearity)
- explicitly store flow on arcs of separator C

will show it suffices to store face labels
for just the faces adjacent to separator C

Flow Representation 2/4

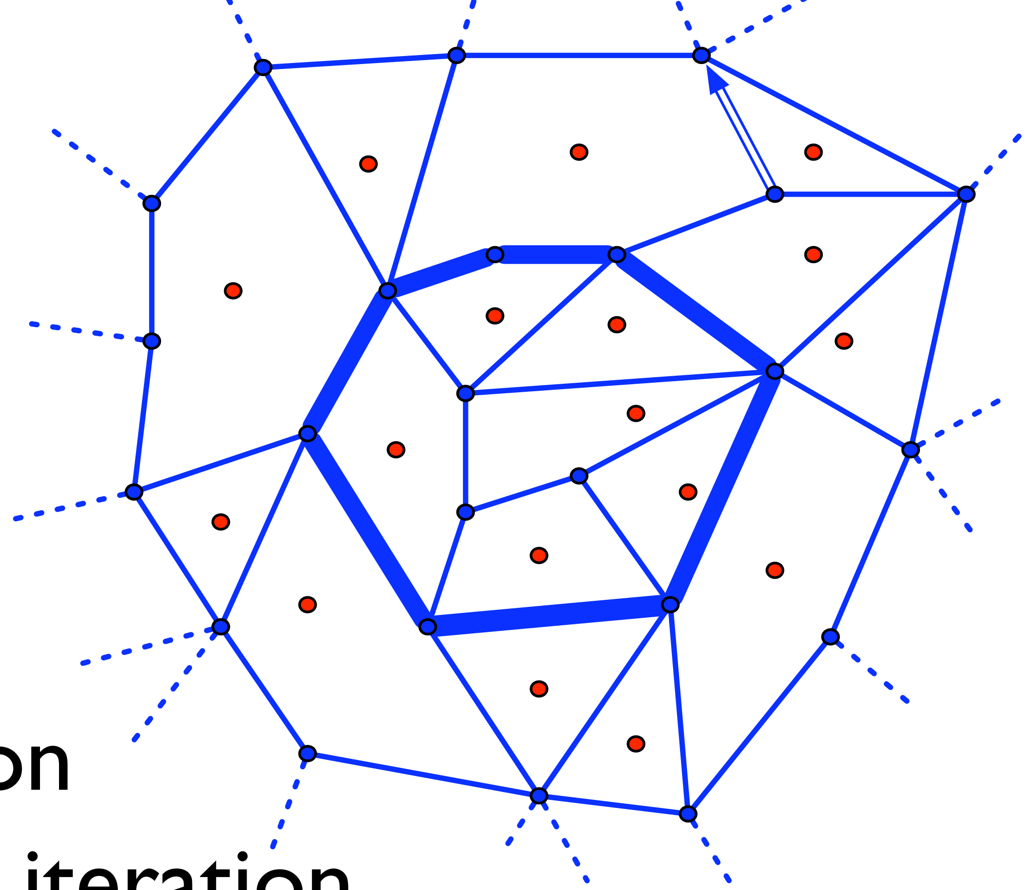


f_0 - flow after recursive calls

f - flow on C's arcs up to current iteration

d - accumulated face labels up to current iteration

Flow Representation 2/4

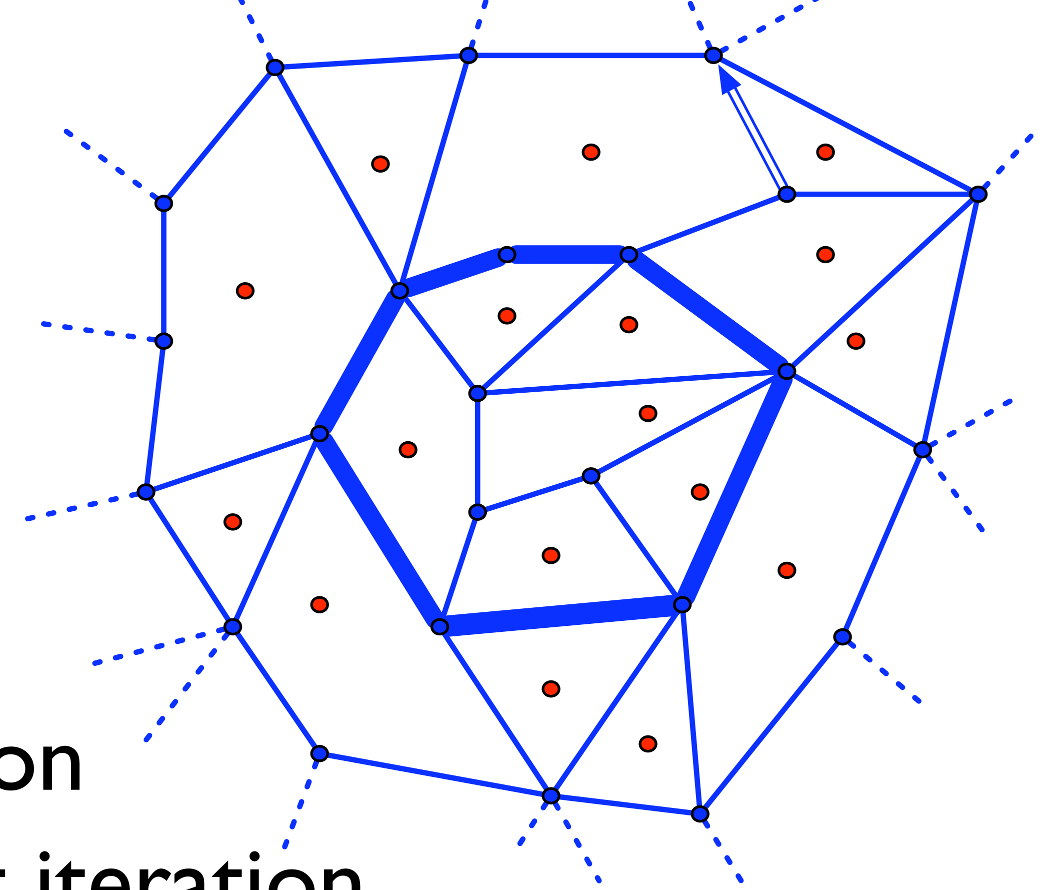


f_0 - flow after recursive calls

f - flow on C's arcs up to current iteration

d - accumulated face labels up to current iteration

Flow Representation 2/4

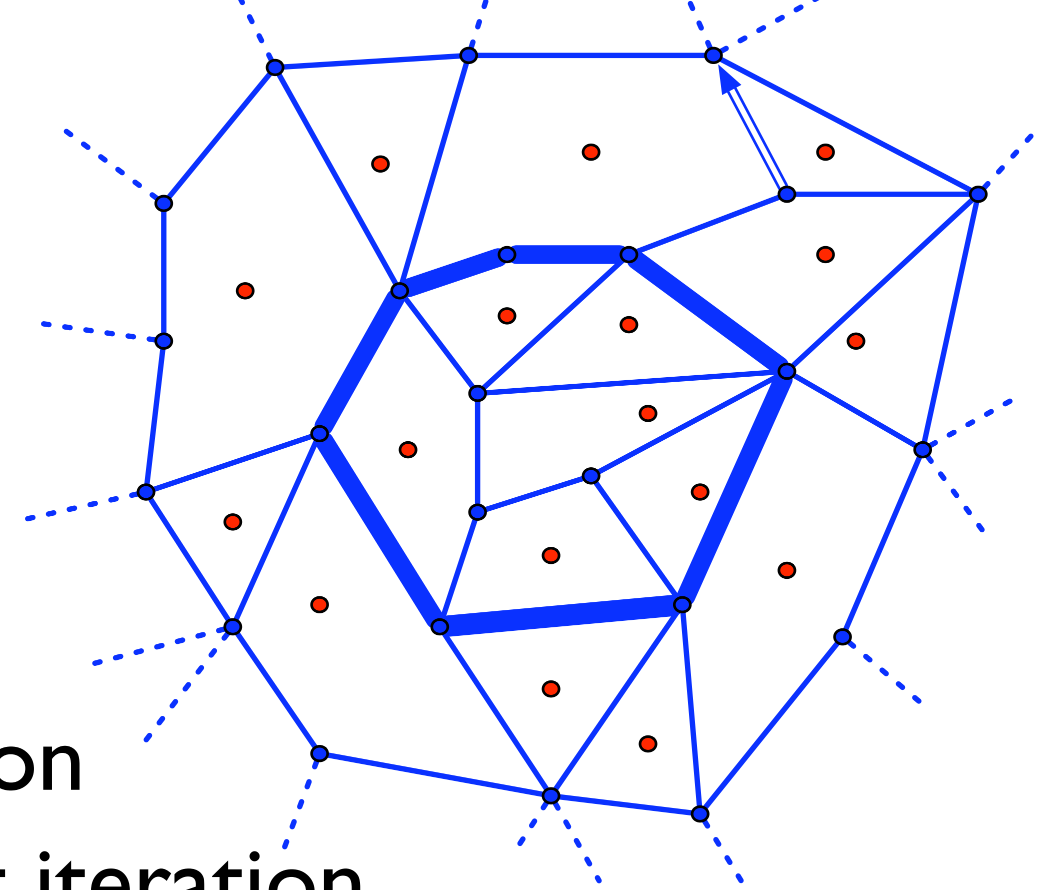


f_0 - flow after recursive calls
f - flow on C's arcs up to current iteration
d - accumulated face labels up to current iteration

- for an arc a not on C, flow is:

  $f_0(a) + d(\text{face right of } a) - d(\text{face left of } a)$

Flow Representation 2/4



f_0 - flow after recursive calls

f - flow on C's arcs up to current iteration

d - accumulated face labels up to current iteration

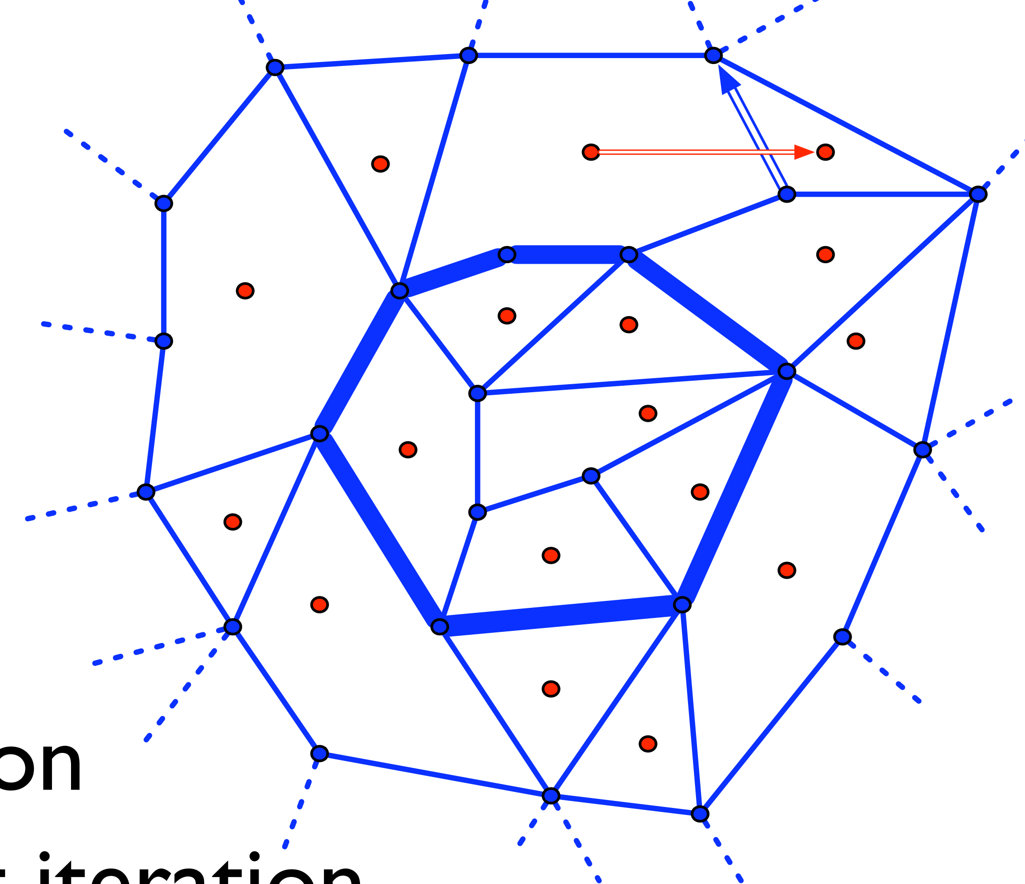
- for an arc a not on C, flow is:

  $f_0(a)$ + d(face right of a) - d(face left of a)

- residual capacity of a is:

  $c(a)$ - $f_0(a)$ - d(face right of a) + d(face left of a)

Flow Representation 2/4



f_0 - flow after recursive calls

f - flow on C's arcs up to current iteration

d - accumulated face labels up to current iteration

- for an arc a not on C, flow is:

  $f_0(a)$ + d(face right of a) - d(face left of a)

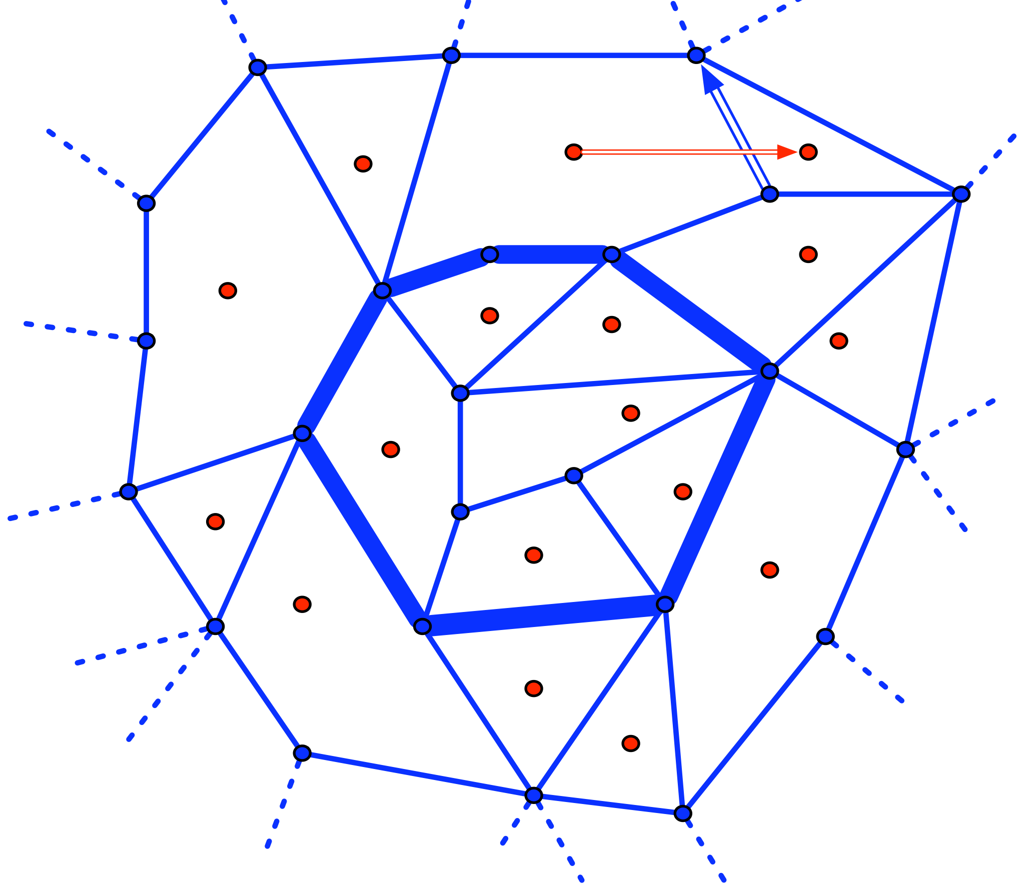
- residual capacity of a is:

  $c(a)$ - $f_0(a)$ - d(face right of a) + d(face left of a)

- length of dual of a is:

  $c(a)$ - $f_0(a)$ - d(head of dual of a) + d(tail of dual of a)

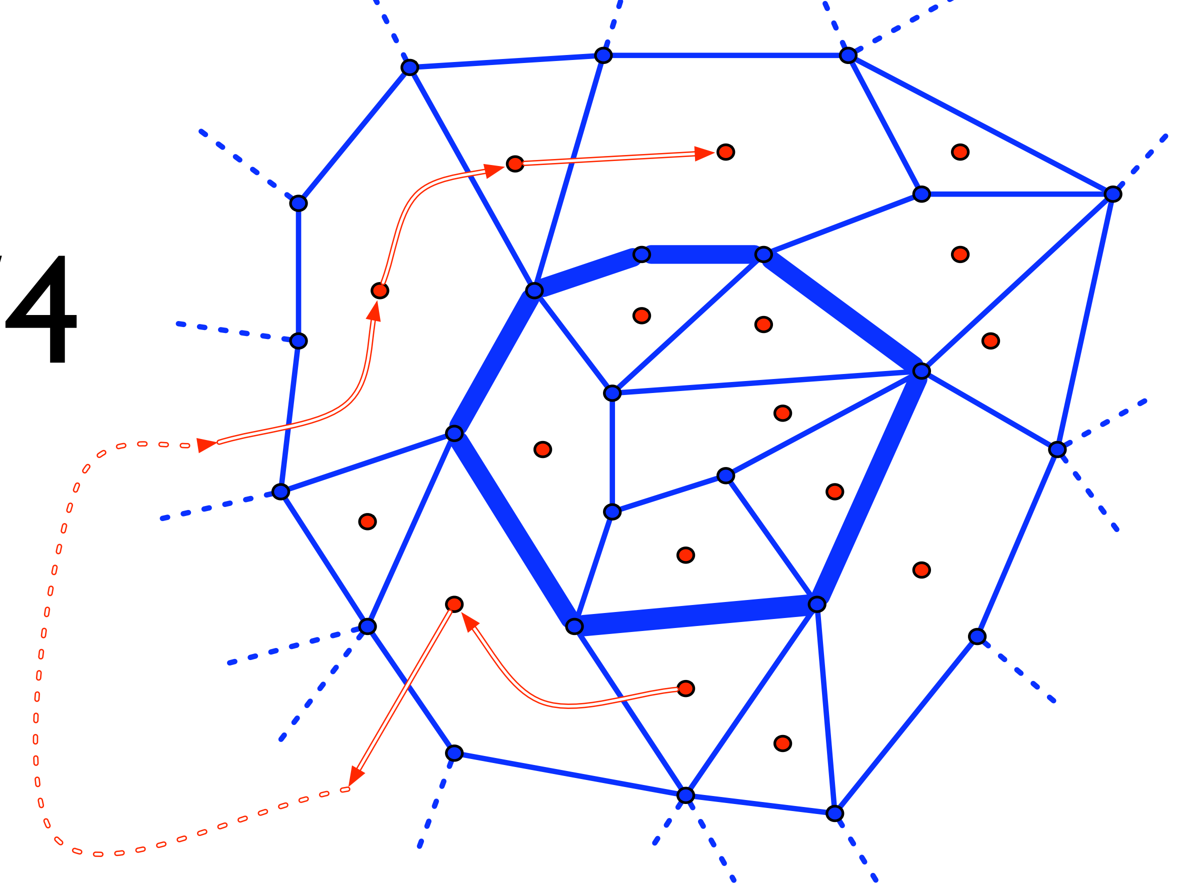
Flow Representation 3/4



- length of dual of a is:

  $c(a) - f_0(a) - d(\text{head of dual of } a) + d(\text{tail of dual of } a)$

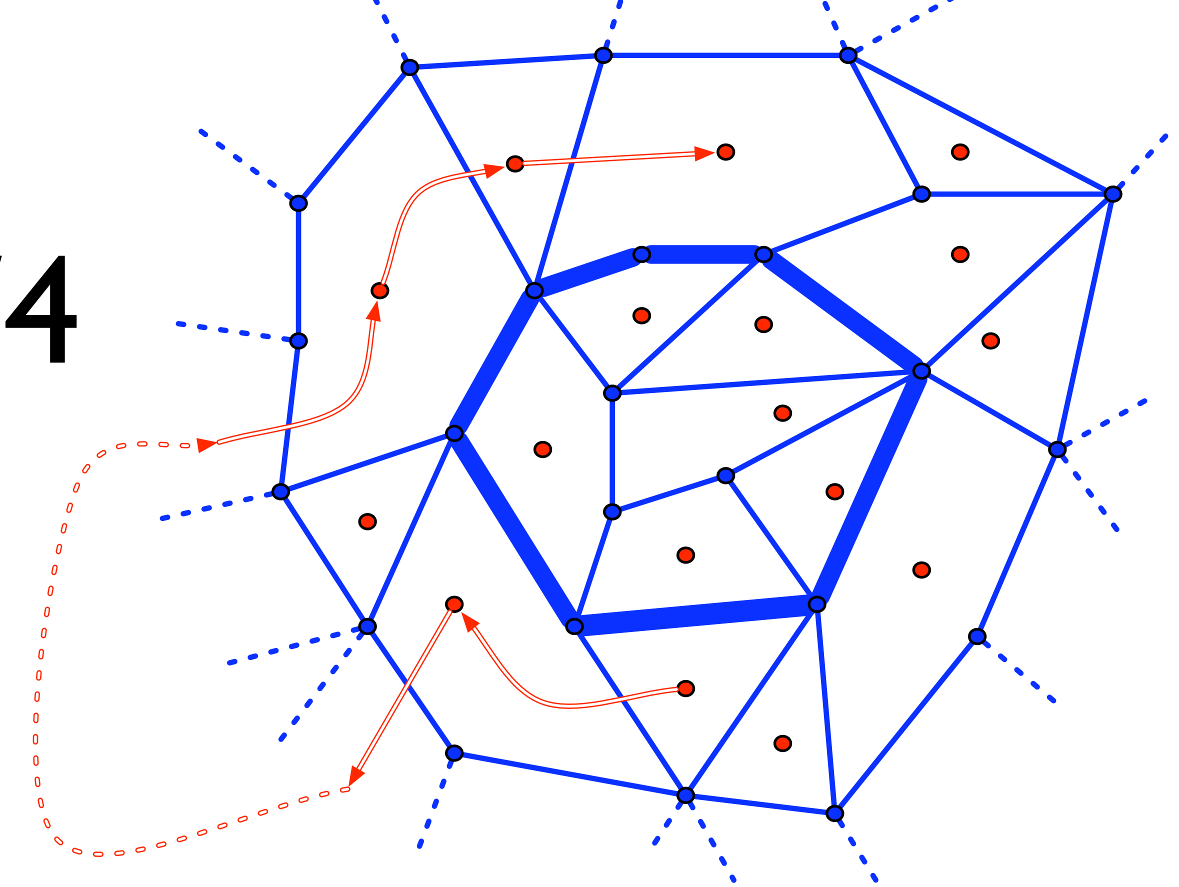
Flow Representation 3/4



- length of dual of a is:

$$c(a) - f_0(a) - d(\text{head of dual of } a) + d(\text{tail of dual of } a)$$

Flow Representation 3/4



- length of dual of a is:

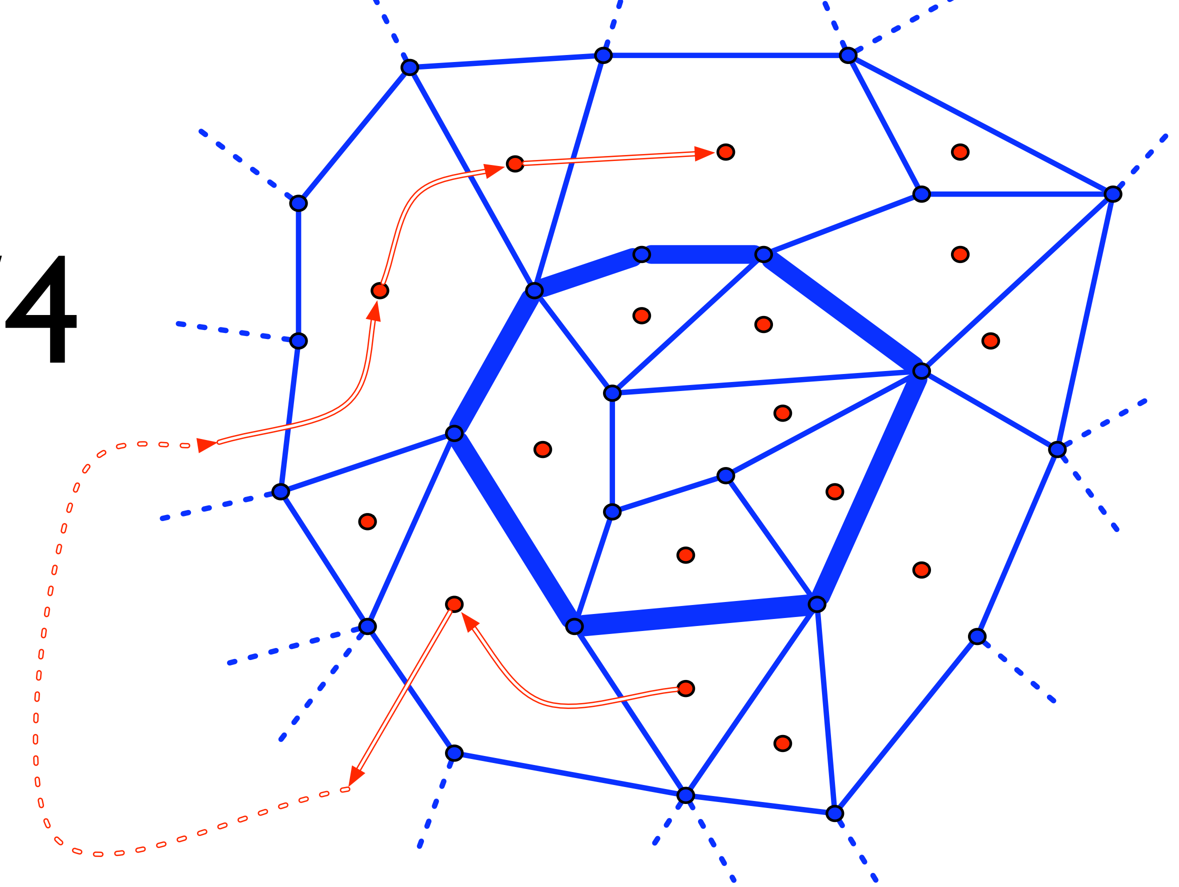
  $c(a) - f_0(a) - d(\text{head of dual of } a) + d(\text{tail of dual of } a)$

- length of any dual path P that does not use dual arcs of C is:

  $\sum c(a) - f_0(a) - d(\text{head of dual of } a) + d(\text{tail of dual of } a)$

  $= d(\text{end of } P) - d(\text{start of } P) + \sum c(a) - f_0(a)$

Flow Representation 3/4



- length of dual of a is:

  $c(a) - f_0(a) - d(\text{head of dual of } a) + d(\text{tail of dual of } a)$

- length of any dual path P that does not use dual arcs of C is:

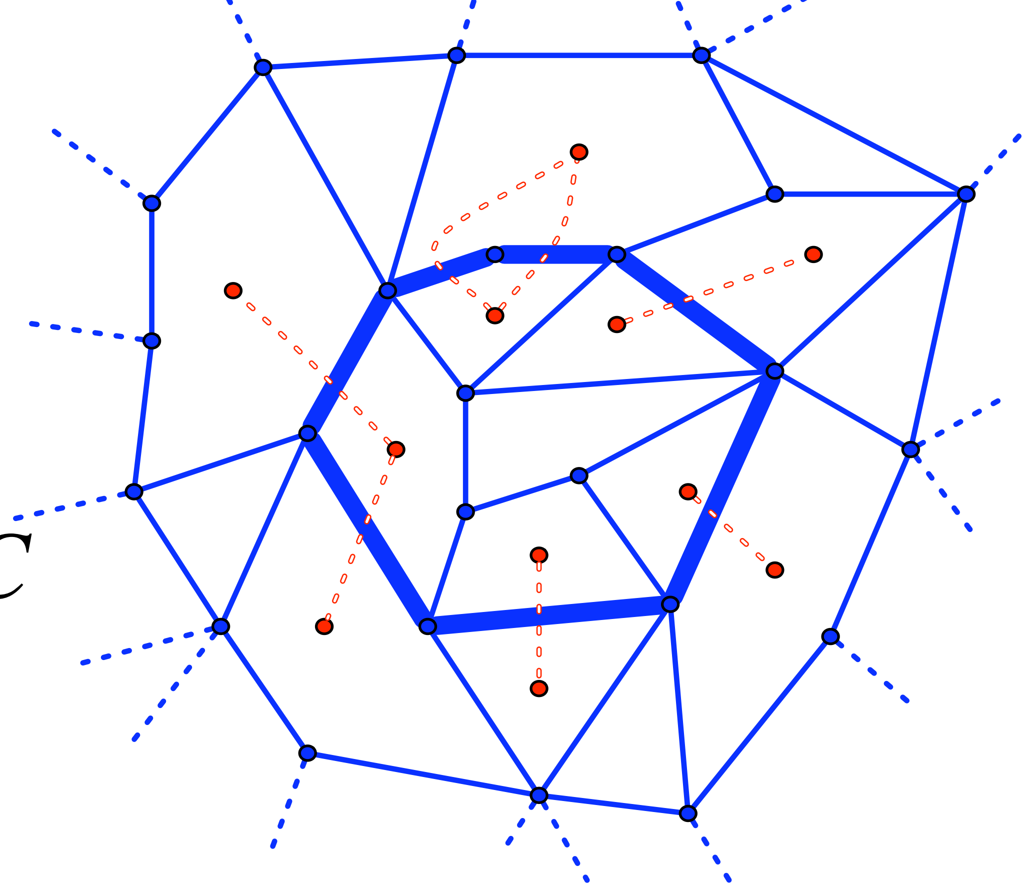
  $\sum c(a) - f_0(a) - d(\text{head of dual of } a) + d(\text{tail of dual of } a)$

  $= d(\text{end of } P) - d(\text{start of } P) + \sum c(a) - f_0(a)$

- ignoring arcs of C, shortest paths are independent of d

  note: length of shortest path does change by $d(\text{end of } P) - d(\text{start of } P)$

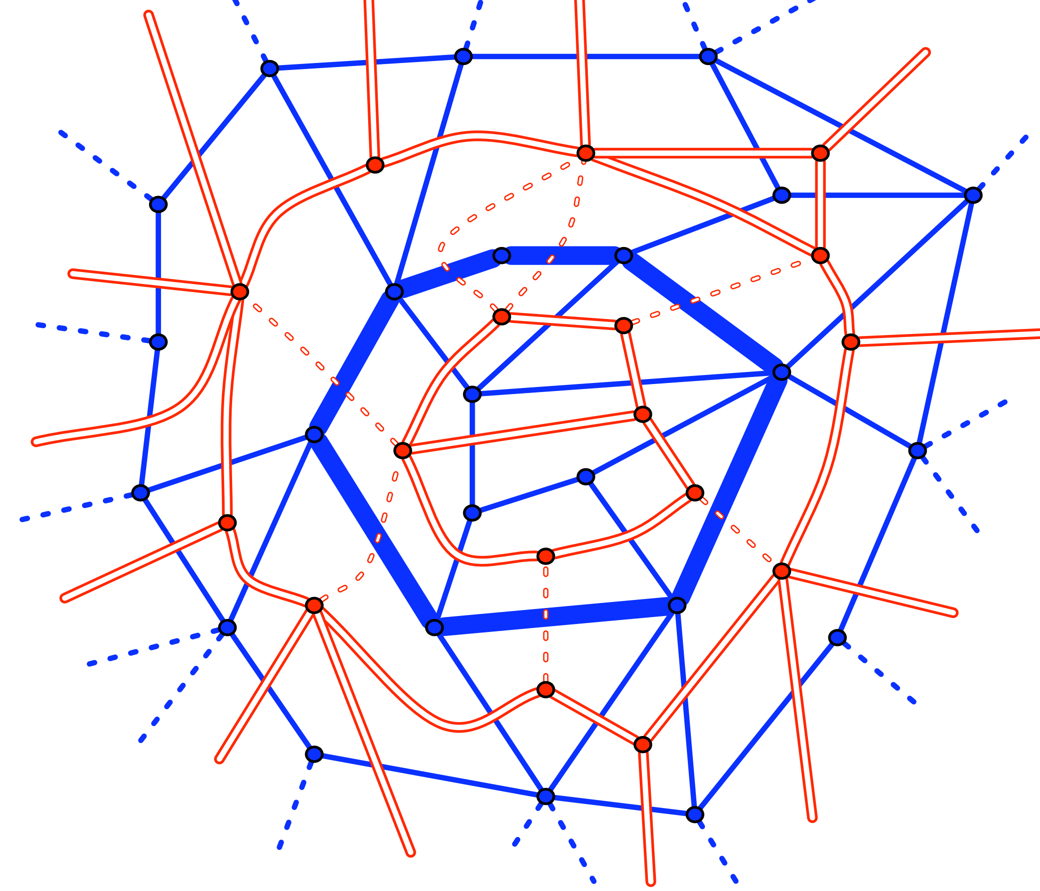
Flow Representation 4/4



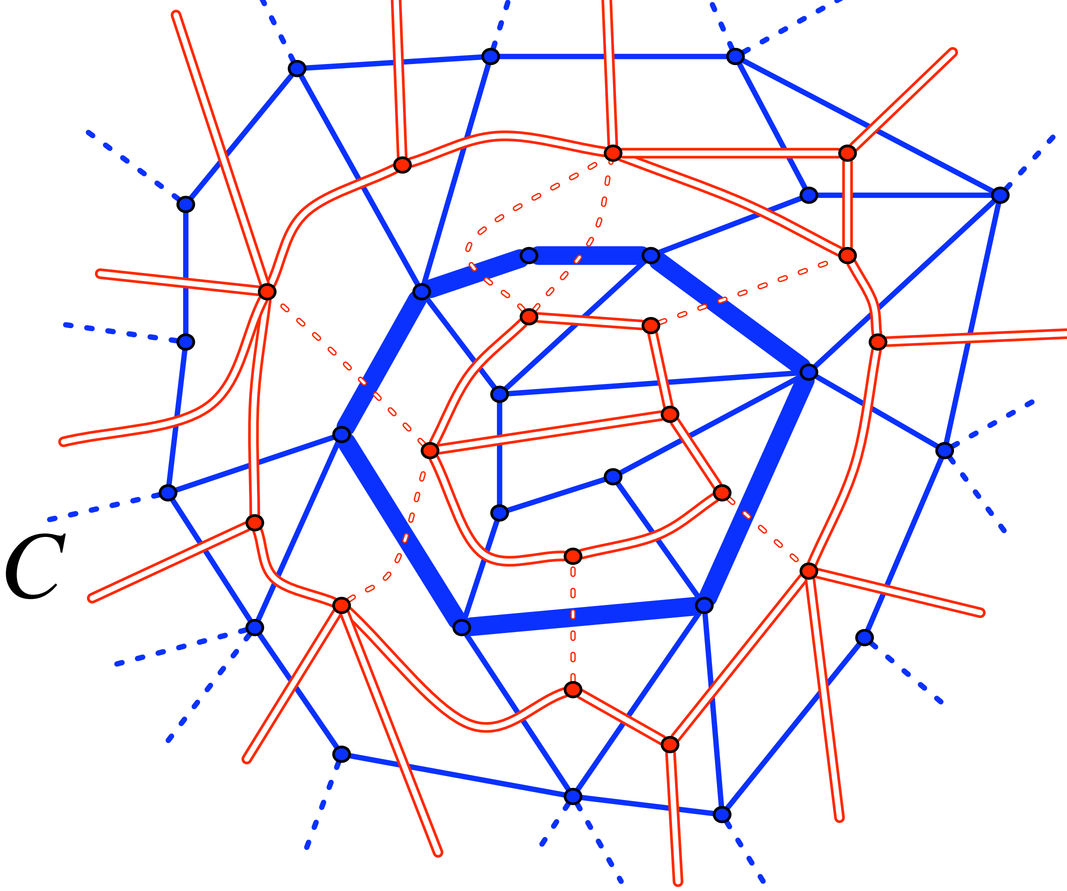
- X = set of faces adjacent to separator C

  = set of endpoints of dual arcs of C

Flow Representation 4/4

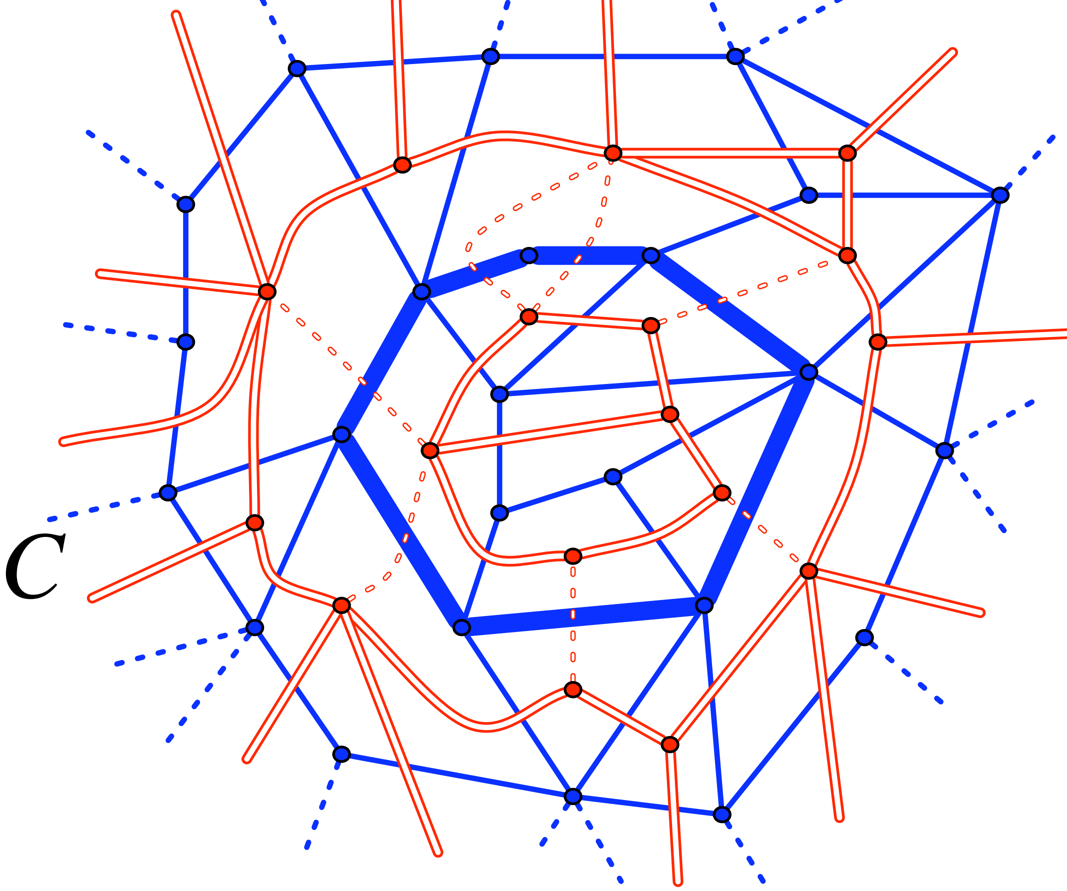


Flow Representation 4/4



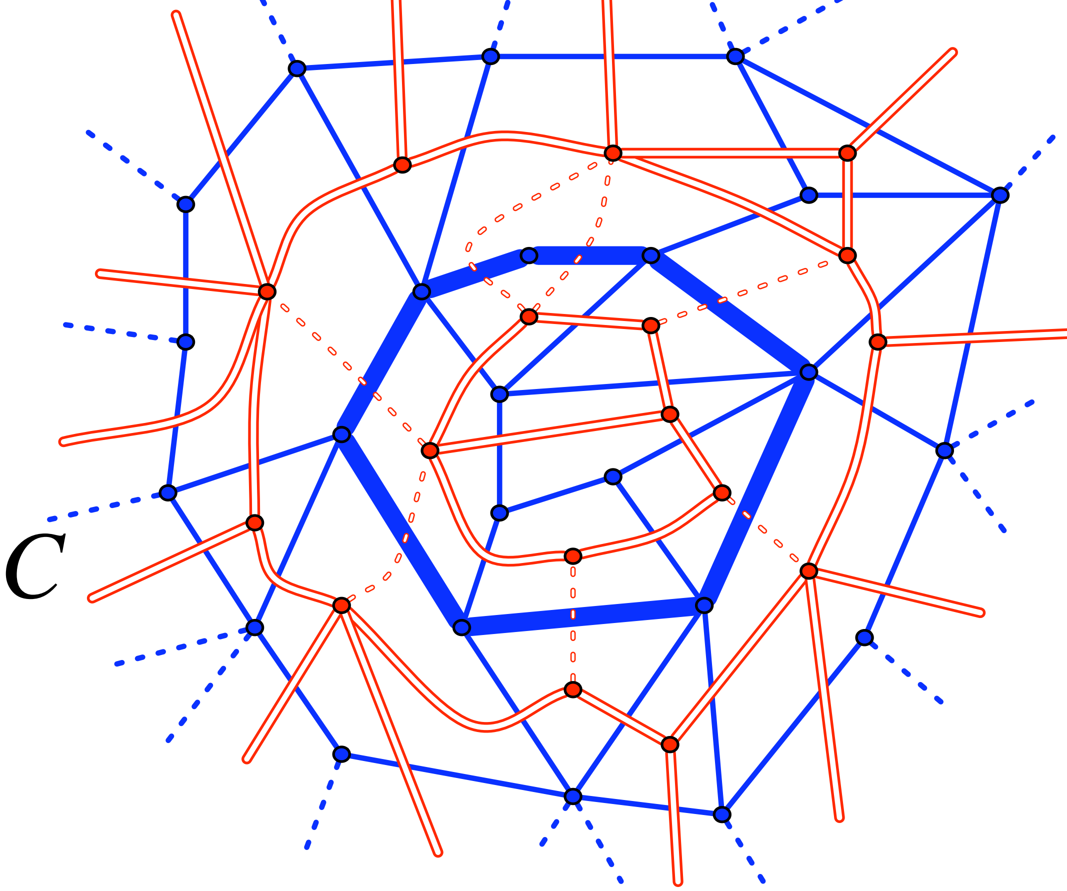
- X = set of faces adjacent to separator C
  = set of endpoints of dual arcs of C
- H - dual graph without dual arcs of C

Flow Representation 4/4



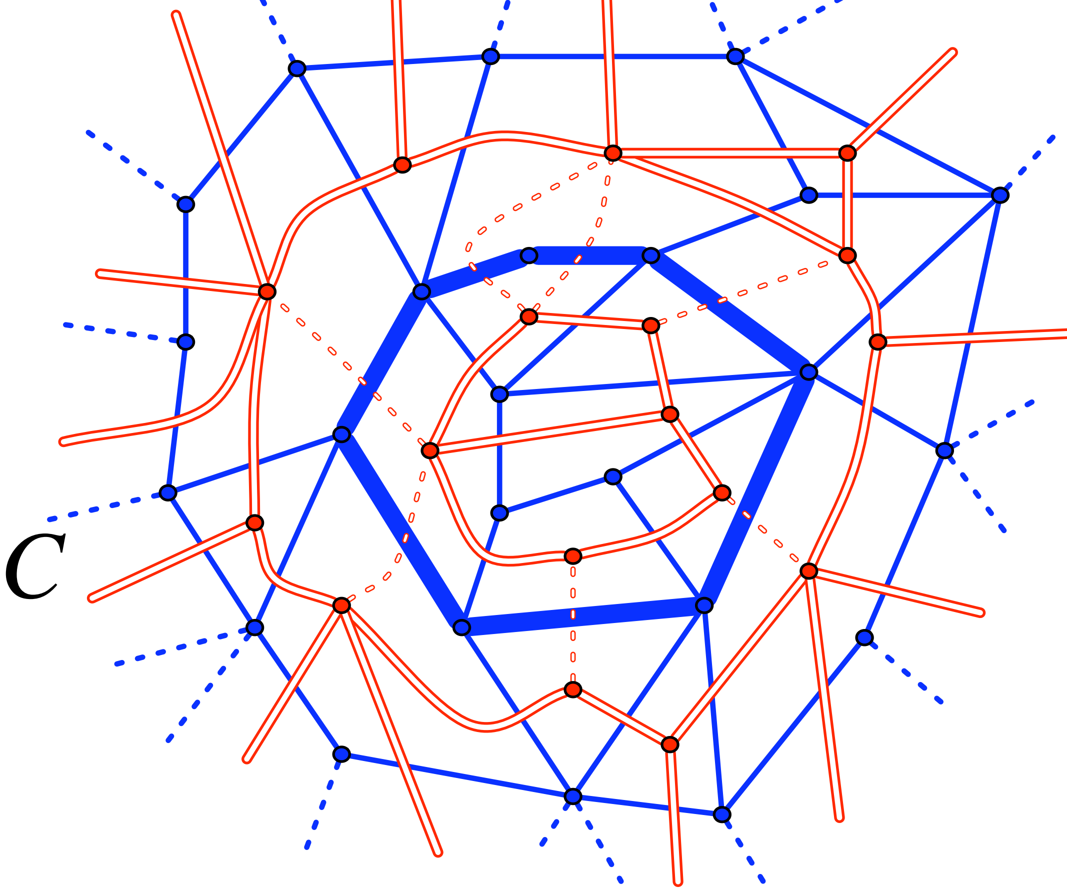
- X = set of faces adjacent to separator C
  = set of endpoints of dual arcs of C
- H - dual graph without dual arcs of C
- any shortest path in dual graph can be decomposed into:
  - shortest paths in H
  - dual arcs of C

Flow Representation 4/4



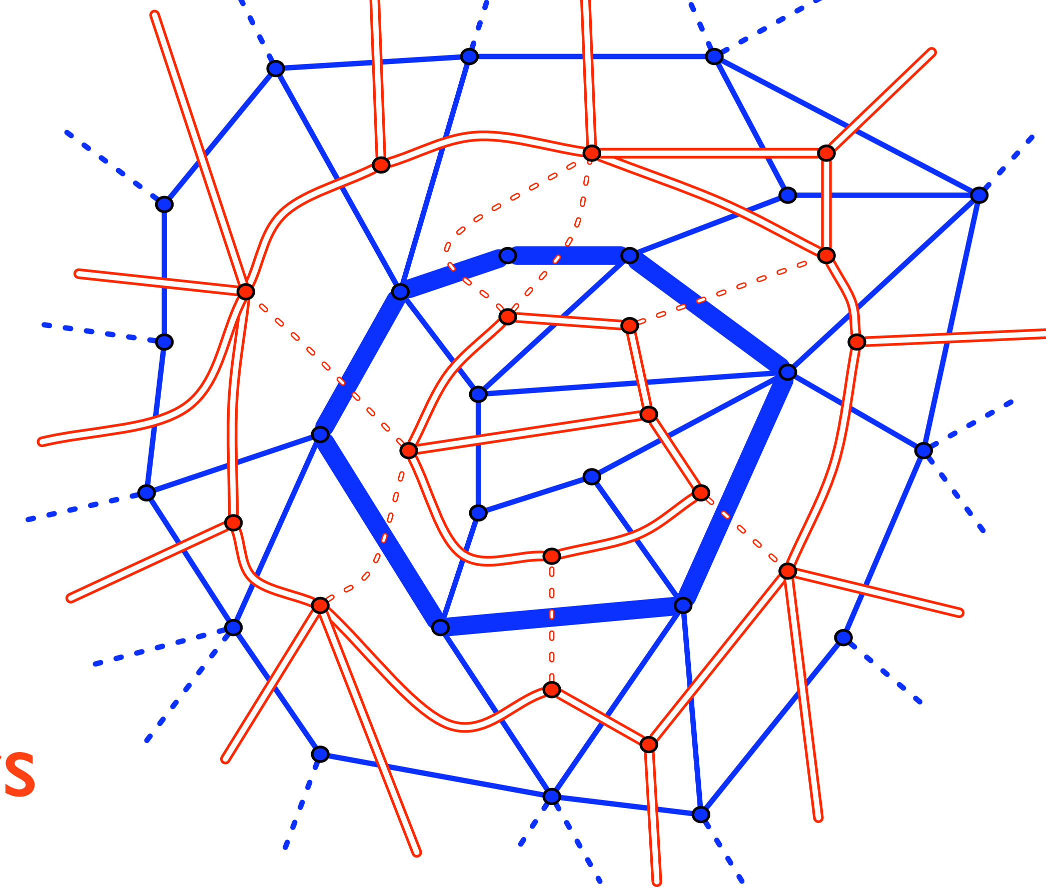
- X = set of faces adjacent to separator C
  = set of endpoints of dual arcs of C
- H - dual graph without dual arcs of C
- any shortest path in dual graph can be decomposed into:
  - shortest paths in H
  - dual arcs of C
- precompute all-pair shortest paths between nodes of X in H
  - can be done in $O(n \log n)$ time [Klein SODA'05]
  - these shortest paths do not change
  - for $x,y \in X$, length of x-to-y path changes by $d(x)$ - $d(y)$

Flow Representation 4/4



- X = set of faces adjacent to separator C
  = set of endpoints of dual arcs of C
- H - dual graph without dual arcs of C
- any shortest path in dual graph can be decomposed into:
  - shortest paths in H
  - dual arcs of C
- precompute all-pair shortest paths between nodes of X in H
  - can be done in $O(n \log n)$ time [Klein SODA'05]
  - these shortest paths do not change
  - for $x,y \in X$, length of x-to-y path changes by $d(x)$ - $d(y)$
- suffices to maintain face labels for X and explicit flow for C

Efficient Implementation



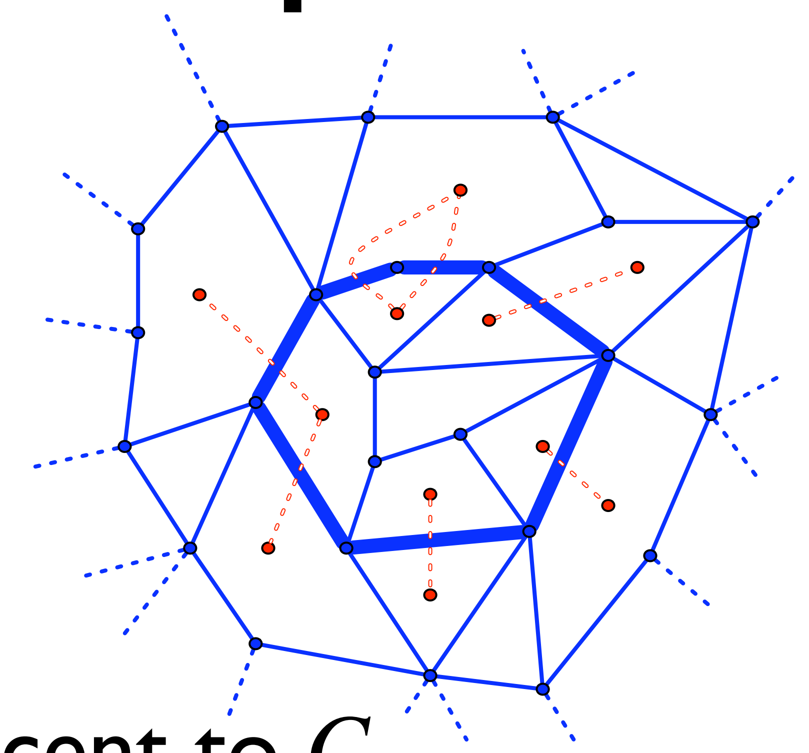
- precompute all-pair shortest paths between nodes of X in H $O(n)$ pairs
- maintain:
  - face labels for X $O(\sqrt{n})$ faces
  - explicit flow for C $O(\sqrt{n})$ arcs
- can implement Dijkstra's algorithm with this representation in $O(\sqrt{n}\log^2 n)$ time using a modification of a data-structure of Fakcharoenphol and Rao [FOCS'01]

running time: $O(\sqrt{n}) \cdot O(\sqrt{n}\log^2 n) = O(n\log^2 n)$

separator nodes

time for max-flow between neighbors using compact representation

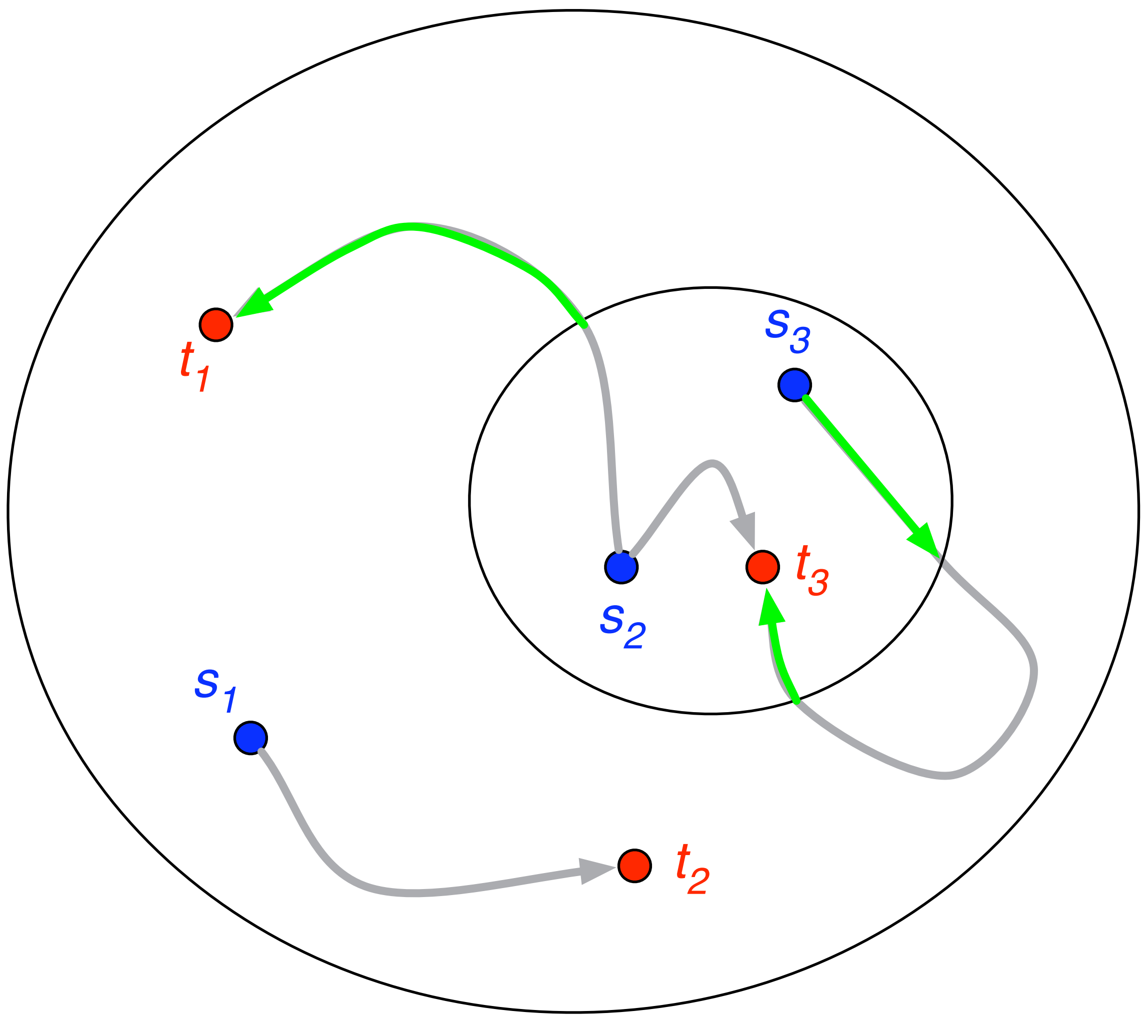
Back to the Entire Graph



- with compact representation we have:
  - explicit flow f on all arcs of C
  - accumulated face labels only for faces adjacent to C
- need to extend face labels to all faces
- can be done using one more shortest-path computation in the dual which takes linear time

Recall High-Level Algorithm

- find separator
- recursive problem (almost): eliminate residual paths
  - from sources to sinks
  - from sources to separator
  - from separator to sinks
- eliminate residual paths from + to - on separator

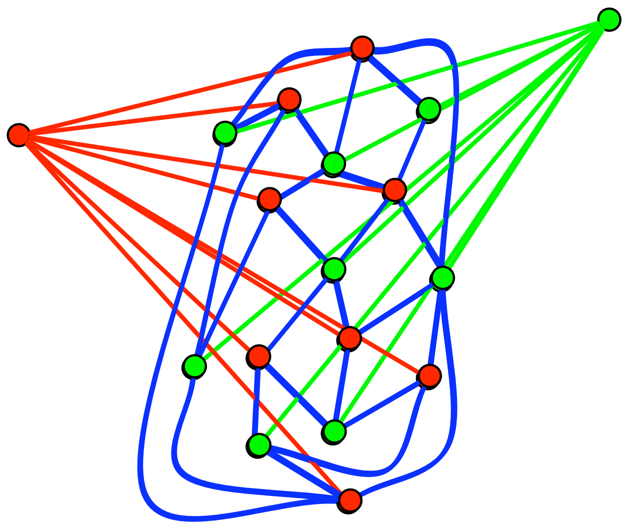


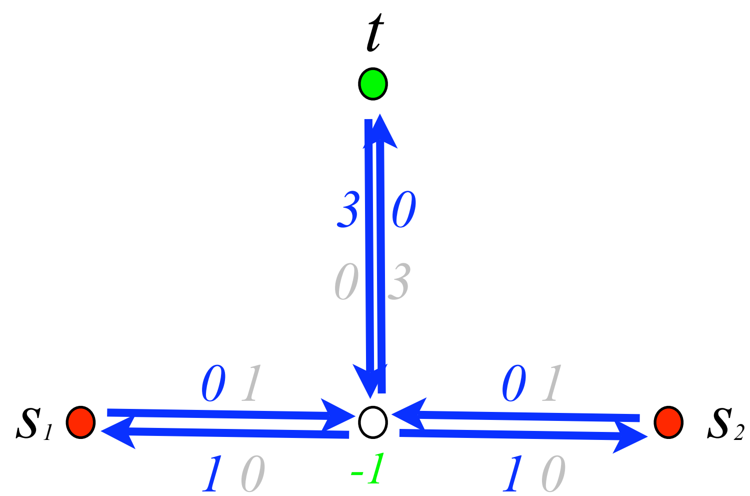
- return flow from + to sources and from sinks to -

running time: $O(n \log^3 n)$

Open Questions/Directions

- can running time be improved?
  (bottleneck is Fakcharoenphol and Rao's data structure and its modification)
- can this technique be adapted to bounded-genus graphs?
- implementation





Thank You!