STATS 207: Time Series Analysis
Autumn 2020

Lecture 16: Seasonal Decomposition, Dynamic Linear Models with Switching

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November 4th 2020

Genera Info

- HW4 is out. Due on Monday 11/16/2020.
- Next Wednesday: Professor David Donoho will talk about "Bootstrap Reality Check & Technical Trading Rules"

Outline

Model Estimation – Example

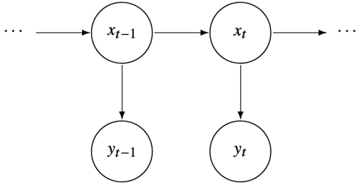
Seasonal Decomposition

Dynamic Linear Models with Switching

State-Space Model (review)

- **State Equation**:

$$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + \Upsilon \boldsymbol{u}_t + \boldsymbol{w}_t,$$



where

  - $\boldsymbol{x}_t$, $\boldsymbol{w}_t$ have dimension p,
  - $\boldsymbol{u}_t$ has dimension r,
  - $\boldsymbol{w}_t \overset{iid}{\sim} \mathcal{N}(0, Q)$.

- **Observation Equation**:

$$\boldsymbol{y}_t = A_t \boldsymbol{x}_t + \Gamma \boldsymbol{u}_t + \boldsymbol{v}_t$$

  - $\boldsymbol{y}_t$, $\boldsymbol{v}_t$ have dimension q,
  - $\boldsymbol{v}_t \overset{iid}{\sim} \mathcal{N}(0, R)$.

- **Initial Conditions**: $\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$.

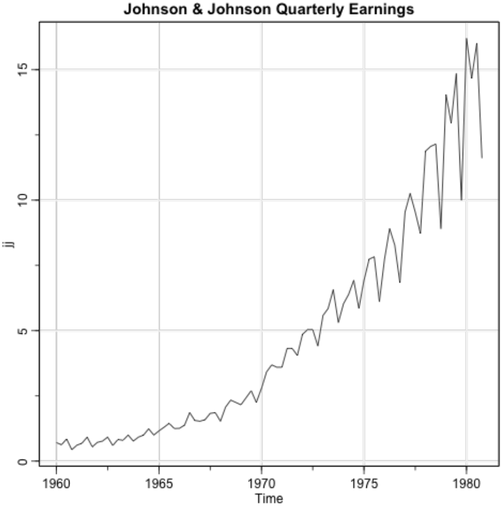
Estimation (review)

Estimation of...

- **State** $\boldsymbol{x}_t$ given data $\boldsymbol{y}_{1:s} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_s\}$ and **model parameters** $\Theta = (\mu_0, \Sigma_0, \Phi, \Gamma, \Upsilon, R, Q)$:
  - **Kalman Filter** ($s = t$) and **Smoother** ($s > t$).
- **Model parameters** $\Theta = (\mu_0, \Sigma_0, \Phi, \Gamma, \Upsilon, R, Q)$ given data $\boldsymbol{y}_{1:n}$:
  - **Maximum Likelihood Estimation** (two methods):
    1. **Likelihood maximization** using Newton Raphson.
    2. **Expectation-Maximization** (EM).

Model Estimation – Example

Example 6.10: Seasonal Decomposition of JnJ Quarterly Earnings



Exponential growth trend + 4 seasonal components

JnJ Seasonal Decomposition, I

- Random **trend** and **seasonal** components in **noise**:

$$y_t = T_t + S_t + v_t,$$

  where:

  - T_t is **trend** increasing **exponentially**: $T_t = \phi T_{t-1} + w_{t1}$,
  - S_t is a **seasonal** component: $S_t + S_{t-1} + S_{t-2} + S_{t-3} = w_{t2}$.
- State-space form ($p = 4$, $q = 1$):
  - Observation Equation:

$$y_t = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_t \\ S_t \\ S_{t-1} \\ S_{t-2} \end{bmatrix} + v_t$$

  - State Dynamics:

$$\begin{bmatrix} T_t \\ S_t \\ S_{t-1} \\ S_{t-2} \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_{t-1} \\ S_{t-1} \\ S_{t-2} \\ S_{t-3} \end{bmatrix} + \begin{bmatrix} w_{t1} \\ w_{t2} \\ 0 \\ 0 \end{bmatrix}$$

  - Covariances:

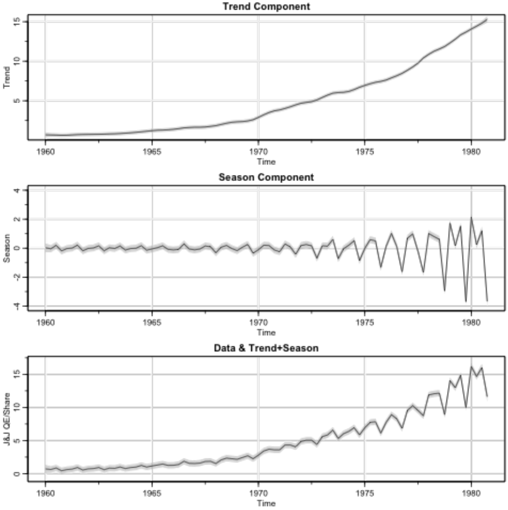
$$Q = \begin{bmatrix} q_{11} & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R = [r_{11}].$$

JnJ Seasonal Decomposition, II

- Parameters to estimate: $\Theta = (\phi, r_{11}, q_{11}, q_{22}, \mu_0, \Sigma_0)$
- Initial conditions:
  - $\phi^{(0)} = 1.03$ (3% annual growth rate)
  - **Initial State mean**: $\mu_0^{(0)} = (.7, 0, 0, 0)'$
  - **Initial State uncertainty**: $\Sigma_0^{(0)} = .04 \cdot I_{4\times 4}$
  - **State covarinace**: $q_{11}^{(0)} = q_{22}^{(0)} = .01$
  - **Measurement error covariance**: $r_{11}^{(0)} = .25$

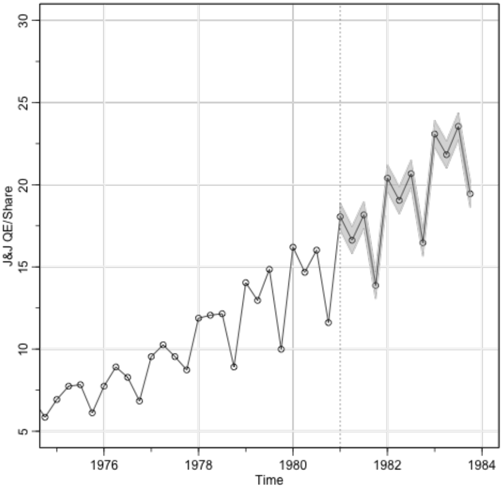
```r
# Initial Parameters
mu0      = c(.7,0,0,0)
Sigma0   = diag(.04,4)
init.par = c(1.03, .1, .1, .5)  # Phi[1,1], the 2 Qs and R

# Function to Calculate Likelihood
Linn=function(para){
 ...}

# Estimation
est = optim(init.par, Linn, NULL, method="BFGS", hessian=TRUE,
                control=list(trace=1,REPORT=1))
SE  = sqrt(diag(solve(est$hessian)))
```

JnJ Seasonal Decomposition, III



JnJ Seasonal Decomposition, IV

Prediction $\pm 2\text{SE}$:



Dynamic Linear Models with Switching

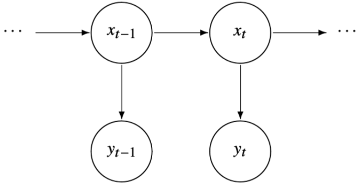
Switching Models

- Starting point is the **same**:
  - **State Equation**:

    $$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + \Upsilon \boldsymbol{u}_t + \boldsymbol{w}_t,$$

  - **Observation Equation**:

    $$\boldsymbol{y}_t = A_t \boldsymbol{x}_t + \Gamma \boldsymbol{u}_t + \boldsymbol{v}_t$$



- **Now assume**:
  - A_t is **not known**.
  - A_t is **one of** m matrices, $\{M_1, \ldots, M_m\}$.

Example 6.19: Tracking Multiple Targets

- Tracking objects without identification:
  - $\boldsymbol{x}_t \in \mathbb{R}^3$: position of **three** targets **fully identified**.
  - $\boldsymbol{y}_t \in \mathbb{R}^3$: position of targets, **not identified**.
  - A_t is a permutation matrix. Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  - Generalization: A_t is a **partial** permutation matrix.
  - **Goal**: infer the true trajectories of the three targets from **noisy, unidentified measurements**.

Example 6.20: Modeling Economic Change, I

Hamilton (1989) et. seq.

- Data are generated by

$$y_t = z_t + n_t,$$

where (z_t) is **AR** and (n_t) is a **random walk** with **switching drift**:

$$n_t = n_{t-1} + \alpha_0 + \alpha_1 S_t, \qquad S_t = \begin{cases} 0 & \text{Recession,} \\ 1 & \text{Growth.} \end{cases}$$

- Suppose z_t is AR(2):

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + w_t,$$

or

$$\nabla y_t = z_t - z_{t-1} + \alpha_0 + \alpha_1 S_t.$$

Example 6.20: Modeling Economic Change, II

State-space model:

- **State Vector**:

$$\boldsymbol{x}_t = (z_t, z_{t-1}, \alpha_0, \alpha_1)'$$

- **Observation Matrices**:

$$M_1 = \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}.$$

- **State Equation**:

$$\begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Observation Equation**:

$$\nabla y_t = A_t \boldsymbol{x}_t + v_t, \qquad A_t \in \{M_1, M_2\}.$$

- **Goal**: Infer the **recession state** (S_t) from **noisy measurements** (∇y_t).

A NEW APPROACH TO THE ECONOMIC ANALYSIS OF NONSTATIONARY TIME SERIES AND THE BUSINESS CYCLE

BY JAMES D. HAMILTON[1]

This paper proposes a very tractable approach to modeling changes in regime. The parameters of an autoregression are viewed as the outcome of a discrete-state Markov process. For example, the mean growth rate of a nonstationary series may be subject to occasional, discrete shifts.

The econometrician is presumed not to observe these shifts directly, but instead must draw probabilistic inference about whether and when they may have occurred based on the observed behavior of the series. The paper presents an algorithm for drawing such probabilistic inference in the form of a nonlinear iterative filter. The filter also permits estimation of population parameters by the method of maximum likelihood and provides the foundation for forecasting future values of the series.

An empirical application of this technique to postwar U.S. real GNP suggests that the periodic shift from a positive growth rate to a negative growth rate is a recurrent feature of the U.S. business cycle, and indeed could be used as an objective criterion for defining and measuring economic recessions. The estimated parameter values suggest that a typical economic recession is associated with a 3% permanent drop in the level of GNP.

KEYWORDS: Switching regression, segmentation, nonstationary, business cycle, nonlinear filtering, regime changes.

Switching Model Ingredients

- Time-varying (**true**) **probabilities**

$$\pi_j(t) = \Pr\left(A_t = M_j\right), \qquad \{1, \dots, m\}, \qquad t = 1, \dots, n.$$

- **Filtered probabilities**:

$$\pi_j(t|s) = \Pr\left(A_t = M_j | \boldsymbol{y}_{1:s}\right)$$

"Time-varying **estimate** of **probability** of being in obs-state j given the **data to time** s".

- **Predicted state**:

$$\boldsymbol{x}_t^{t-1} = \mathbb{E}\left[\boldsymbol{x}_t | \boldsymbol{y}_{1:t-1}\right], \qquad P_t^{t-1} = \mathrm{Var}\left(\boldsymbol{x}_t | \boldsymbol{y}_{1:t-1}\right),$$

- **Filtered state**:

$$\boldsymbol{x}_t^{t} = \mathbb{E}\left[\boldsymbol{x}_t | \boldsymbol{y}_{1:t}\right], \qquad P_t^{t} = \mathrm{Var}\left(\boldsymbol{x}_t | \boldsymbol{y}_{1:t}\right).$$

Switching Model Recursions

- **Prediction**:

$$\boldsymbol{x}_t^{t-1} = \Phi \boldsymbol{x}_{t-1}^{t-1}, \qquad P_t^{t-1} = \Phi P_{t-1}^{t-1} \Phi' + Q.$$

- **Innovations** and **Kalman Gain** **for every** $j = 1, \ldots, m$:

$$\epsilon_{tj} = \boldsymbol{y}_t - M_j \boldsymbol{x}_t^{t-1}, \qquad \Sigma_{tj} = M_j P_t^{t-1} M_j' + R,$$

$$K_{tj} = P_t^{t-1} M_j' \Sigma_{tj}^{-1}.$$

- **Update**:

$$\boldsymbol{x}_t^t = \boldsymbol{x}_t^{t-1} + \sum_{j=1}^{m} \pi_j(t|t) K_{tj} \epsilon_{tj}, \qquad P_t^t = \sum_{j=1}^{m} \pi_j(t|t)(I - K_{tj} M_j) P_t^{t-1},$$

Derivation of Switching Model Recursions

$$\boldsymbol{x}_t^t = \mathbb{E}\left[\boldsymbol{x}_t | \boldsymbol{y}_{1:t}\right] = \mathbb{E}\left[\mathbb{E}\left[\boldsymbol{x}_t | \boldsymbol{y}_{1:t}, A_t\right] | \boldsymbol{y}_{1:t}\right] \ldots$$

$$= \mathbb{E}\left[\sum_{j=1}^{m} \mathbb{E}\left[\boldsymbol{x}_t | \boldsymbol{y}_{1:t}, A_t = M_j\right] 1(A_t = M_j) | \boldsymbol{y}_{1:t}\right]$$

$$= \mathbb{E}\left[\sum_{j=1}^{m} (\boldsymbol{x}_t^{t-1} + K_{tj}\boldsymbol{\epsilon}_{tj}) 1(A_t = M_j) | \boldsymbol{y}_{1:t}\right]$$

$$= \sum_{j=1}^{m} \mathbb{E}\left[1(A_t = M_j) | \boldsymbol{y}_{1:t}\right] (\boldsymbol{x}_t^{t-1} + K_{tj}\boldsymbol{\epsilon}_{tj})$$

$$= \sum_{j=1}^{m} \pi_j(t|t)(\boldsymbol{x}_t^{t-1} + K_{tj}\boldsymbol{\epsilon}_{tj})$$

$$= \boldsymbol{x}_t^{t-1} + \sum_{j=1}^{m} \pi_j(t|t) K_{tj}\boldsymbol{\epsilon}_{tj}$$

Derivation of Probability Filters

- Possible models for $\pi_j(t) = \Pr(A_t = M_j)$:
  (I) **IID**:
$$\pi_j(t) = \pi_j, \qquad j = 1, \ldots, m.$$
  (II) **Markov**:
$$\pi_{ij} = \Pr(A_t = M_i | A_{t-1} = M_j), \qquad j, i = 1, \ldots, m$$
- Case (I) is simple:
$$\pi_j(t|t) = \frac{\pi_j(t)\mathbf{P}_j(t|t-1)}{\sum_{k=1}^{m} \pi_k(t)\mathbf{P}_k(t|t-1)},$$
  by Bayes rule, where
$$\begin{aligned}\mathbf{P}_j(t|t-1) &\equiv \mathbf{f}(\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1}, A_t = M_j) \\ &\equiv f_{\boldsymbol{y}_t|\boldsymbol{y}_{1:t-1}, A_t}(\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1}, M_j)\end{aligned}$$
  is the conditional **density** of $\boldsymbol{y}_t$ given that RVs
  $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{t-1} =$ (observed) $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{t-1}$ and RV $A_t = M_j$.
- Case (II) is next...

Derivation of Probability Filters – Markov Case

Example 6.21: **Assume** (A_t) is a **Markov Chain** Model with transition probabilities $\pi_{ij} = \Pr(A_t = M_i | A_{t-1} = M_j)$.

Recursion for $\pi_j(t|t)$:

$$\pi_j(t|t) = \Pr(A_t = M_j | \boldsymbol{y}_{1:t}) = \frac{\mathbf{f}(A_t = M_j, \boldsymbol{y}_t | \boldsymbol{y}_{1:t-1})}{\mathbf{f}(\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1})}$$

$$= \frac{\Pr(A_t = M_j | \boldsymbol{y}_{1:t-1}) \mathbf{f}(\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1}, A_t = M_j)}{\sum_{k=1}^{m} \Pr(A_t = M_k | \boldsymbol{y}_{1:t-1}) \mathbf{f}(\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1}, A_t = M_k)}$$

$$= \frac{\pi_j(t|t-1) \mathbf{P}_j(t|t-1)}{\sum_{k=1}^{m} \pi_k(t|t-1) \mathbf{P}_k(t|t-1)}$$

$$\pi_j(t|t-1) = \Pr(A_t = M_j | \boldsymbol{y}_{1:t-1})$$

$$= \sum_{i=1}^{m} \Pr(A_t = M_j, A_{t-1} = M_i | \boldsymbol{y}_{1:t-1})$$

$$= \sum_{i=1}^{m} \Pr(A_t = M_j | A_{t-1} = M_i) \Pr(A_{t-1} = M_i | \boldsymbol{y}_{1:t-1})$$

$$= \sum_{i=1}^{m} \pi_{ij} \cdot \pi_i(t-1|t-1)$$

Conditional Densities Recursion

- To evaluate $\mathbf{P}_j(t|t-1)$, we must run over **all m^{t-1} possible histories** of $A_1, \ldots, A_{t-1}$.
- For each $\ell \in \{1, \ldots, m^{t-1}\}$ match $J(\ell) \in \{1, \ldots, m\}^{t-1}$. Define the event

$$A_{(t-1)} = M_{(\ell)} \Leftrightarrow \left\{A_1 = M_{J(\ell)_1}, \ldots, A_{t-1} = M_{J(\ell)_{t-1}}\right\}$$

- We have:

$$\mathbf{P}_j(t|t-1) = \mathbf{f}(\boldsymbol{y}_t | A_t = M_j, \boldsymbol{y}_{1:t-1})$$

$$= \sum_{\ell=1}^{m^{t-1}} \Pr\left(A_{(t-1)} = M_{(\ell)} | \boldsymbol{y}_{1:t-1}\right) \mathbf{f}\left(\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1}, A_t = M_j, A_{(t-1)} = M_{(\ell)}\right)$$

$$\triangleq \sum_{\ell=1}^{m^{t-1}} \alpha(\ell) \mathcal{N}\left(\boldsymbol{y}_t | \mu_{tj}(\ell), \Sigma_{tj}(\ell)\right),$$

where $\mathcal{N}(y|\mu, \Sigma)$ is the corresponding **normal density**, and

$$\mu_{tj}(\ell) = \mathbb{E}\left[\boldsymbol{y}_t | \boldsymbol{y}_{1:t-1}, A_t = M_t, A_{(t-1)} = M_{(\ell)}\right] = M_j \boldsymbol{x}_t^{t-1}(\ell),$$

$$\Sigma_{tj}(\ell) = M_j P_t^{t-1}(\ell) M_j' + R.$$

Conditional Density Approximation

- Conditional density

$$\mathbf{P}_j(t|t-1) = \sum_{\ell=1}^{m^{t-1}} \alpha(\ell)\mathcal{N}\left(\boldsymbol{y}_t|\mu_{tj}(\ell), \Sigma_{tj}(\ell)\right)$$

- A Gaussian mixture with one term for each **possible history** $M(\ell)$.
- **Impossible complexity** of m^{t-1} histories.
- Proposal: **Normal Approximation** by

$$\mathbf{P}_j(t|t-1) \approx \mathcal{N}(\boldsymbol{y}_t|M_j\boldsymbol{x}_t^{t-1}, \Sigma_{tj})$$

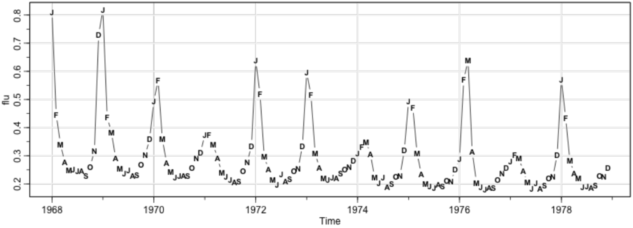
(density with the same mean and std as the full distribution)

- Complete-Data Log-Likelihood:

$$\log L_Y(\Theta) = \sum_{t=1}^{n} \log(\mathbf{f}(\boldsymbol{y}_t|\boldsymbol{y}_{1:t-1})) = \sum_{t=1}^{n} \log\left(\sum_{j=1}^{m} \pi_j(t|t-1)\mathbf{P}_j(t|t-1)\right).$$

Example 6.22: Analysis of Influenza Data, I

- (y_t) is U.S. monthly pneumonia and influenza deaths per 10,000.



- Propose a **structural model with three components**:
  - **Seasonal**: $x_{t1} = \alpha_1 x_{t-1,1} + \alpha_2 x_{t-2,1} + w_{t1}$
  - **Epidemic**: $x_{t2} = \beta_0 + \beta_1 x_{t-1,2} + w_{t2}$
  - **Trend**: $x_{t3} = x_{t-1,3} + w_{t3}$
- **Observed Process**:

$$y_t = x_{t1} + x_{t2} I_t(\text{epidemic}) + x_{t3} + v_t$$

Example 6.22: Analysis of Influenza Data, II

State-space with switching formulation:

- **State Dynamics**:

$$\overbrace{\begin{bmatrix} x_{t1} \\ x_{t-1,1} \\ x_{t2} \\ x_{t3} \end{bmatrix}}^{\boldsymbol{x}_t} = \overbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{\Phi} \begin{bmatrix} x_{t-1,1} \\ x_{t-2,1} \\ x_{t-1,2} \\ x_{t-1,3} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \beta_0 \\ 0 \end{bmatrix}}^{\Upsilon \boldsymbol{u}_t} + \overbrace{\begin{bmatrix} w_{t1} \\ 0 \\ w_{t2} \\ w_{t3} \end{bmatrix}}^{\boldsymbol{w}_t}.$$

- **Observation Equation**:

$$y_t = A_t \boldsymbol{x}_t + v_t, \qquad A_t = \begin{cases} M_1 = [1, 0, 0, 1] & \text{no epidemic,} \\ M_2 = [1, 0, 1, 1] & \text{epidemic.} \end{cases}$$

- **Switching Model**: (A_t) is a Markov chain

$$\pi_{11} = \pi_{22} = .75, \qquad \pi_{12} = \pi_{21} = .25.$$

Example 6.22: Analysis of Influenza Data, III

- Use Newton-Raphson to maximize **approximate** log-likelihood.
- Initialization: $\pi_1(1|0) = \pi_2(1|0) = 1/2$.
- Fitting Result:

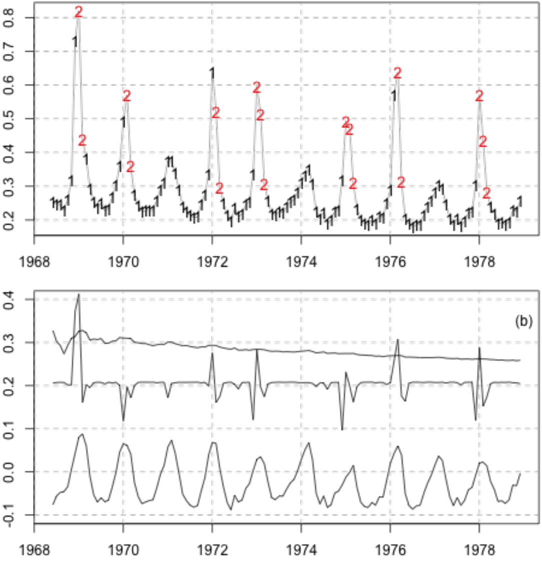
Table 6.3. *Estimation Results for Influenza Data*

Parameter	Initial Model Estimates	Final Model Estimates
α_1	1.422 (.100)	1.406 (.079)
α_2	−.634 (.089)	−.622 (.069)
β_0	.276 (.056)	.210 (.025)
β_1	−.312 (.218)	—
σ_1	.023 (.003)	.023 (.005)
σ_2	.108 (.017)	.112 (.017)
σ_v	.002 (.009)	—

Estimated standard errors in parentheses

- β_1 and σ_v are not significant in initial model.
- Remove these parameters to get the final model.

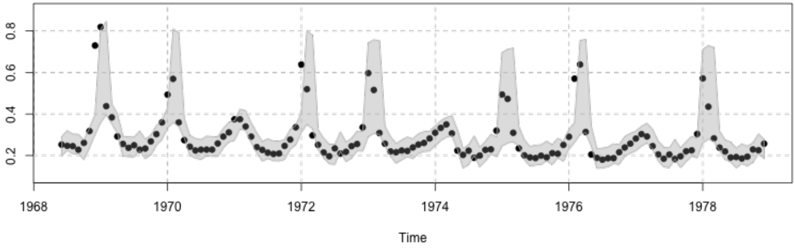
Example 6.22: Analysis of Influenza Data, IV



Example 6.22: Analysis of Influenza Data, V

One-month-ahead predictions:

$$\hat{y}_t^{t-1} = \begin{cases} M_1 \hat{\boldsymbol{x}}_t^{t-1}, & \hat{\pi}_1(t|t-1) > \hat{\pi}_2(t|t-1), \\ M_2 \hat{\boldsymbol{x}}_t^{t-1}, & \hat{\pi}_1(t|t-1) \leq \hat{\pi}_2(t|t-1). \end{cases}$$



Example 6.20: Modeling Economic Change, I (again)

Hamilton (1989) et. seq.

- Data are generated by

$$y_t = z_t + n_t,$$

where (z_t) is AR and (n_t) is a random walk with **switching drift**:

$$n_t = n_{t-1} + \alpha_0 + \alpha_1 S_t, \qquad S_t = \begin{cases} 0 & \text{Recession,} \\ 1 & \text{Growth.} \end{cases}$$

- Suppose z_t is AR(2):

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + w_t,$$

or

$$\nabla y_t = z_t - z_{t-1} + \alpha_0 + \alpha_1 S_t.$$

Example 6.20: Modeling Economic Change, II

Switching State-space model:

- **State Vector**:

$$\boldsymbol{x}_t = (z_t, z_{t-1}, \alpha_0, \alpha_1)$$

- **Observation Matrices**:

$$M_1 = \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}.$$

- **State Equation**:

$$\begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Observation Equation**:

$$\nabla y_t = A_t \boldsymbol{x}_t + v_t, \qquad A_t \in \{M_1, M_2\}.$$

- **Transition probabilities**:

$$\Pr(S_t = 0 | S_t = 0) = q, \qquad Pr(S_t = 1 | S_t = 1) = p.$$

A NEW APPROACH TO THE ECONOMIC ANALYSIS OF NONSTATIONARY TIME SERIES AND THE BUSINESS CYCLE

BY JAMES D. HAMILTON[1]

This paper proposes a very tractable approach to modeling changes in regime. The parameters of an autoregression are viewed as the outcome of a discrete-state Markov process. For example, the mean growth rate of a nonstationary series may be subject to occasional, discrete shifts.

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KEYWORDS: Switching regression, segmentation, nonstationary, business cycle, nonlinear filtering, regime changes.

Hamilton 1989 – Model Estimates

TABLE I

MAXIMUM LIKELIHOOD ESTIMATES OF PARAMETERS AND ASYMPTOTIC STANDARD ERRORS BASED ON DATA FOR U.S. REAL GNP, $t = 1952:\text{II}$ TO $1984:\text{IV}$

Parameter	Estimate	Standard error
α_1	1.522	0.2636
α_0	−0.3577	0.2651
p	0.9049	0.03740
q	0.7550	0.09656
σ	0.7690	0.06676
ϕ_1	0.014	0.120
σ_2	−0.058	0.137
ϕ_3	−0.247	0.107
ϕ_4	−0.213	0.110

Hamilton 1989 – Inferred Recession Periods, I

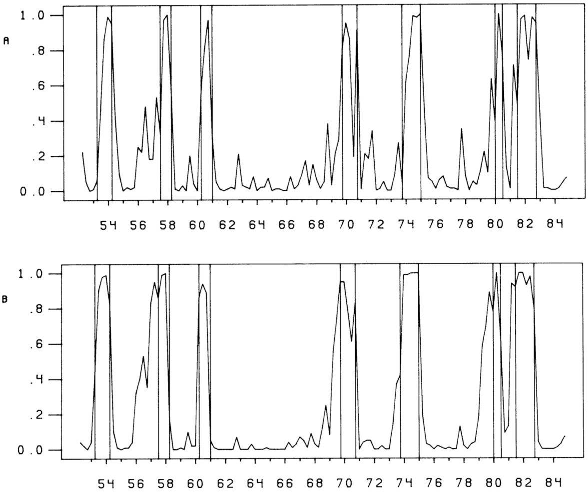

FIGURE 1.—Inferred probability that $S_t = 0$.

Panel (A) reports the inferred probability that the economy was in the falling GNP state at date t using information available at the time $(P[S_t = 0 | y_t, y_{t-1}, \ldots])$. Panel (B) reports the inferred probability that the economy was in the falling GNP state at date t using information available 4 quarters later $(P[S_t = 0 | y_{t+4}, y_{t+3}, \ldots])$.

Inferred Recession Periods, II

ALTERNATIVE DATING OF U.S. BUSINESS CYCLE PEAKS AND TROUGHS AS DETERMINED BY (1) NBER, AND (2) PROBABILITY OF BEING IN RECESSION GREATER THAN 0.5 AS DETERMINED FROM FULL-SAMPLE SMOOTHER

NBER		Smoother	
Peak	Trough	Peak	Trough
1953 : III	1954 : II	1953 : III	1954 : II
1957 : III	1958 : II	1957 : I	1958 : I
1960 : II	1961 : I	1960 : II	1960 : IV
1969 : IV	1970 : IV	1969 : III	1970 : IV
1973 : IV	1975 : I	1974 : I	1975 : I
1980 : I	1980 : III	1979 : II	1980 : III
1981 : III	1982 : IV	1981 : II	1982 : IV

Recap

- Seasonal Decomposition
- DLM with Switching:
    - Extension of Kalman filter recursion
    - Recursions for filtered obs-state
    - Approximation for conditional data density