STATS 207: Time Series Analysis
Autumn 2020

Lecture 11: Spectral Analysis

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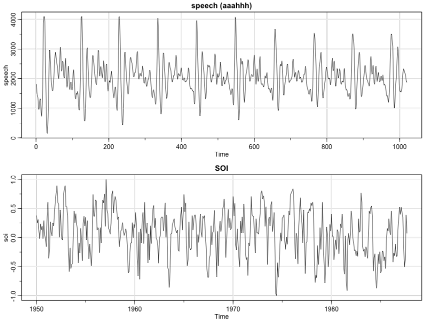
Genera Info

- HW2 is due today.
- HW3 is out. Due Monday 11/2/2020.
- Last chance to submit **midquarter feedback** (anonymously): `https://canvas.stanford.edu/courses/123058/quizzes/84145` (Canvas)
- Additional guest lectures (Prophet, bootstrap)

Motivation



Idea: Use periodic variations to model series

$$x_t = \sum_{k=1}^{q} [U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t)]$$

Outline

Sinusoidal Regression and Periodogram

Sinusoid in Noise (Review)

- $x_t = A\cos(2\pi\omega t + \phi) + w_t$, where
  - A is the **amplitude**.
  - ϕ is the **phase**.
  - ω is the **frequency index** or the **angular velocity**.
- **Linearization** trick

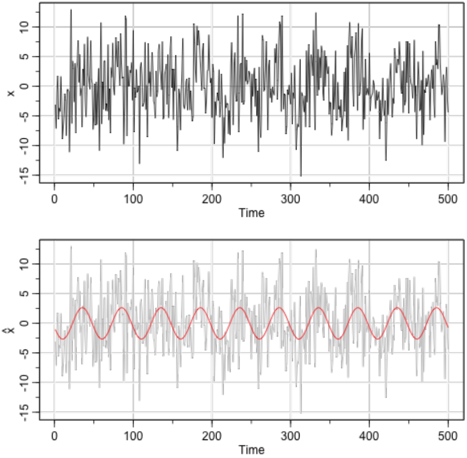
$$\beta_1 \cos(2\pi\omega t) + \beta_2 \sin(2\pi\omega t) = A\cos(2\pi\omega t + \phi)$$

- Fit using cos and sin (instead of A and ϕ):

$$x_t = \beta_1 \cos(2\pi\omega t) + \beta_2 \sin(2\pi\omega t) + w_t.$$

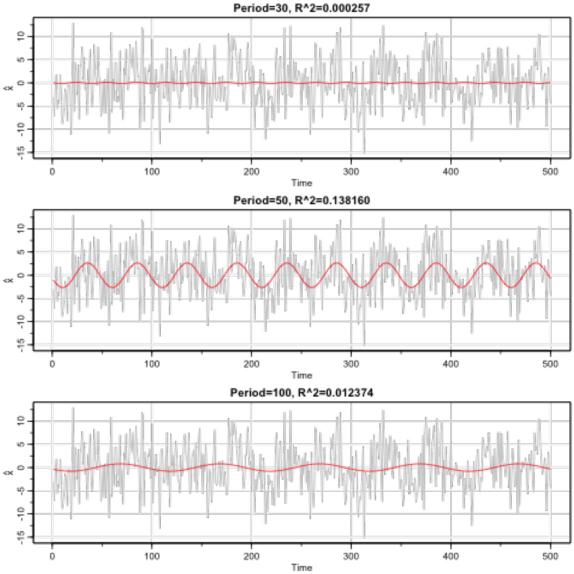
Example 2.10: Signal Hidden in Noise (Review)

```
set.seed(1000)  # so you can reproduce these results
x = 2*cos(2*pi*1:500/50 + .6*pi) + rnorm(500,0,5)
z1 = cos(2*pi*1:500/50)
z2 = sin(2*pi*1:500/50)
summary(fit <- lm(x~0+z1+z2))  # zero to exclude the intercept
par(mfrow=c(2,1)); tsplot(x); tsplot(x, col=8, ylab=expression(hat(x)))
lines(fitted(fit), col=2)
```



How to Determine Periodicity? (Review)

- By Trial and Error:



How to Determine Periodicity? (Review)

- **OLS regression** coefficients

$$\hat{\beta}_1(j/n) = \frac{2}{n}\sum_{t=1}^{n} x_t \cos(2\pi jt/n), \qquad \hat{\beta}_2(j/n) = \frac{2}{n}\sum_{t=1}^{n} x_t \sin(2\pi jt/n)$$

- Measure of **power** in fitted model at frequency $\omega = j/n$:

$$P(j/n) \equiv \hat{\beta}_1^2(j/n) + \hat{\beta}_2^2(i/n)$$

- R^2 at frequency j:

$$R^2 = \frac{P(j/n)}{\sum_{i=1}^{n} P(j/n)}$$

Periodogram, I (Review)

- Discrete Fourier Transform (aka Fast Fourier Transform):

$$d(j/n) \equiv \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i t j/n}, \quad j = 0, \ldots, n-1, \qquad i = \sqrt{-1}.$$

- Computable in O(n log(n)) flops. Standard in digital signal processing.
- Definition: **Periodogram**

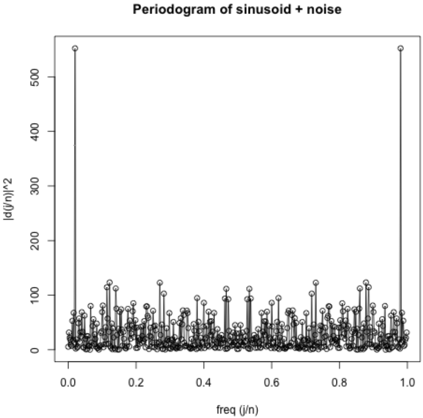
$$I_n(j/n) \equiv |d(j/n)|^2 .$$

- Measure of **power in fitted model** at frequency $\omega = j/n$:

$$P(j/n) = \frac{4}{n} I_n(j/n).$$

Periodogram, II (Review)

```
n = 500
x = 2*cos(2*pi*1:n/50 + .6*pi) + rnorm(n,0,5)
s = fft(x)/sqrt(n)
freq = (0:(n-1) )/n
plot(freq, abs(s)^2, ylab="|d(j/n)|^2", type="ol",
xlab='freq (j/n)', main='Periodogram of sinusoid + noise')
```



Properties if Periodogram I

- **Nonnegativity**

$$I_n(j/n) \geq 0, \quad j = 0, \ldots, n-1.$$

- **Decomposition of variance**: Let n be odd and set $m = (n-1)/2$.

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2 = \frac{1}{m}\sum_{j=1}^{m} I_n(j/n)$$

  ("Pythagorean theorem" or "Parseval's identity").

- Fraction of variance **explained** by sinusoids

$$R^2(j/n) = \frac{I_n(j/n)}{m\hat{\sigma}^2}.$$

- **In words**: "Periodogram indicates the **component of data variance explainable by sinusoids** at frequency j". "This can **never be negative**". "It leverages to the **total variance** of the signal".

Properties of Priodogram II

- Let (w_t) be Gaussian White Noise. Assume n is odd.

$$I_n(j/n) \overset{iid}{\sim} \mathrm{Exp}(1), \quad j = 1, \ldots, (n-1)/2,$$

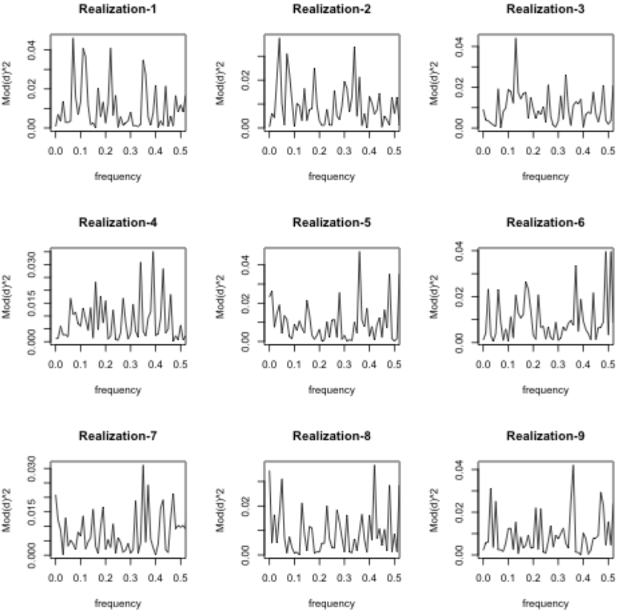
  where $\mathrm{Exp}(\mu)$ is the exponential distribution with mean μ.

$$X \sim \mathrm{Exp}(\mu) \Leftrightarrow F_X(t) = \Pr(X \leq t) = 1 - e^{-t/\mu}, \quad t \geq 0.$$

  In words: "Periodogram of a **Gaussian** white Noise is an **Exponential** White Noise".

- **Mirroring effect**: $I(j/n) = I(1 - j/n)$, $j = 0, \ldots, n-1$.

Periodogram of White Noise – Several Realizations



Spectral Density

Example 4.4: A Periodic Stationary Process

- Consider

$$x_t = w_1 \cos(2\pi\omega_0 t) + w_2 \sin(2\pi\omega_0 t), \qquad w_1, w_2 \overset{iid}{\sim} \mathcal{N}(0, \sigma^2).$$

- Every **realization** of the process is **periodic** with period $1/\omega_0$.
- We have

$$\begin{aligned}\gamma_x(t+h, t) &= \frac{\sigma^2}{2}\cos(2\pi\omega_0(t+h))\cos(2\pi\omega_0 t) \\ &\qquad + \frac{\sigma^2}{2}\sin(2\pi\omega_0(t+h))\sin(2\pi\omega_0 t) \\ &= \sigma^2\cos(2\pi\omega_0 h) = \frac{\sigma^2}{2}e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2}e^{2\pi i\omega_0 h} = \gamma_x(h)\end{aligned}$$

- Write

$$\gamma_x(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} dF(\omega), \qquad F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2 & \omega \ge \omega_0. \end{cases}$$

- Definition: $F(\omega)$ is the **spectral distribution function**.

Spectral Density I

- Property 4.1: If (x_t) is stationary, there exists a unique monotonic function $F(\omega)$, called the **spectral distribution function**, with $F(-\infty) = F(-1/2) = 0$, $F(\infty) = F(1/2) = \gamma(0)$, and

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} dF(\omega)$$

- Property 4.2: If $\gamma(h)$ (of a stationary process (x_t)) satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} f(\omega) d\omega, \qquad h = 0, \pm 1, \pm 2, \ldots,$$

where

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h}, \quad -1/2 \leq \omega \leq 1/2.$$

- Definition: $f(\omega)$ is the **spectral density function** of (x_t).

Spectral Density II

Property 4.2: If $\gamma(h)$ (of a stationary process (x_t)) satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h} f(\omega) d\omega, \qquad h = 0, \pm 1, \pm 2, \ldots,$$

where

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h}, \quad -1/2 \le \omega \le 1/2.$$

In words:

- "If the **covariance function** $\gamma(h)$ is **absolutely summable**, then the **spectral distribution** $F(\omega)$ is **absolutely continuous**. The **spectral density** $f(\omega)$ is the density of the **spectral distribution**.
- "The **spectral density** $f(\omega)$ has a **Fourier series** representation with coefficients given by the **covariance function** $\gamma(h)$".

Properties of Spectral Density, I

- **Nonnegativity**:

$$f_x(\omega) \geq 0, \quad \omega \in (-1/2, 1/2).$$

- Decomposition of variance

$$\mathrm{Var}(x_t) = \gamma_x(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(\omega) d\omega = F_x(1/2).$$

- **In words**: "Spectrum is the **variance explainable by sinusoids** at frequency ω". "This can **never be negative**", "It sums to the **total variance** of the stochastic process".

Relation to Periodogram

- Suppose (x_t) is **stationary** with **absolutely summable** $\gamma_x(h)$.

$$f_x(\omega) = \lim_{n\to\infty} \mathbb{E}\left[I_n\left(\lfloor n\omega \rfloor / n\right)\right], \qquad \omega \in (0, 1/2).$$

- Suppose that x_t is also a **Gaussian** process. Then, approximately

$$I_n(j/n) \overset{approx}{\sim} \mathrm{Exp}\left(f_x(j/n)\right), \qquad j = 0, 1, \ldots, n/2.$$

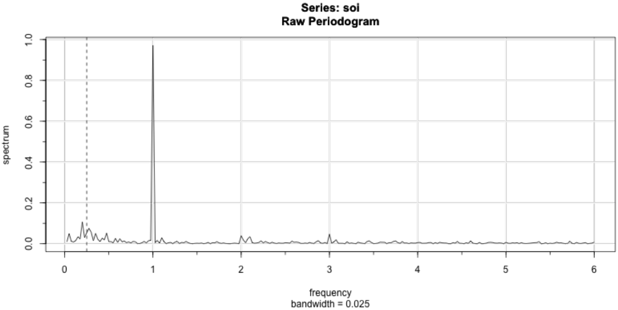
  Equivalently (Property 4.6),

$$\frac{2I_n(j/n)}{f_x(j/n)} \overset{approx}{\sim} \chi_2^2, , \qquad j = 0, 1, \ldots, n/2.$$

- **In words**: "Spectrum gives **typical size** of random variable **periodogram**."

Example 4.13: Priodogram of SOI

```
# n = 480, Delta = 1/12
par(mfrow=c(2,1))
soi.per = mvspec(soi, log="no"); abline(v=1/4, lty=2)
```



Significant power at $\omega = 1\Delta$ and $\omega = \Delta/4$, where $\Delta = 1/12$.

Example 4.13: Priodogram of SOI (cont'd)

From

$$\frac{2I_n(j/n)}{f_x(j/n)} \overset{approx}{\sim} \chi_2^2,$$

approximate $100(1-\alpha)\%$ confidence interval for $f_{SOI}(\omega)$ is found by

$$\frac{2I_n(\lfloor n\omega \rfloor/n)}{\chi_2^2(1-\alpha/2)} \leq f_{SOI}(\omega) \leq \frac{2I_n(\lfloor n\omega \rfloor/n)}{\chi_2^2(\alpha/2)}.$$

```
# Values of SOI's periodogoram at peaks:
soi.per$spec[40]  # 0.97223;  soi pgram at freq 1/12 = 40/480
soi.per$spec[10]  # 0.05372;  soi pgram at freq 1/48 = 10/480

al = .05
 # conf intervals -  returned value:
U = qchisq(al/2,2)    # 0.05063
L = qchisq(1-al/2,2)  # 7.37775
2*soi.per$spec[40]/L  # 0.26355
2*soi.per$spec[40]/U  # 38.40108
2*soi.per$spec[10]/L  # 0.01456
2*soi.per$spec[10]/U  # 2.12220
```

Cannot establish significance of peak at $\omega = \Delta/4$!

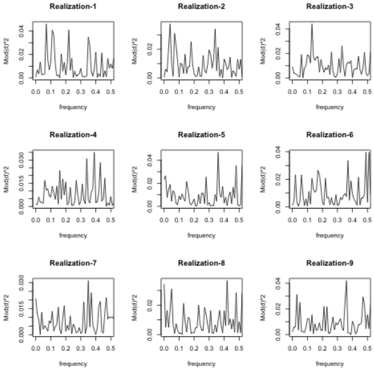
Next Lecture: Better estimate by **smoothing the periodogram**.

Properties of the Spectral Density, II

- If (w_t) is Gaussian white noise, then

$$f_w(\omega) = \sigma_w^2, \quad \omega \in (-1/2, 1/2).$$

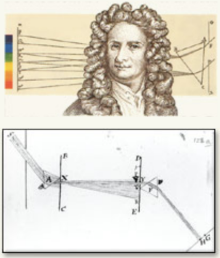
- **In words**: "In a white noise..."
  - "**all ordinates** of the periodogram have the **same expectation**".
  - "the **expectation is the variance** of the process".
  - "**all frequencies** are **present** in equal intensity".



Newton & Spectrum, I

Newton and the Color Spectrum

Our modern understanding of light and color begins with Isaac Newton (1642-1726) and a series of experiments that he publishes in 1672. He is the first to understand the rainbow — he refracts white light with a prism, resolving it into its component colors: red, orange, yellow, green, blue and violet.

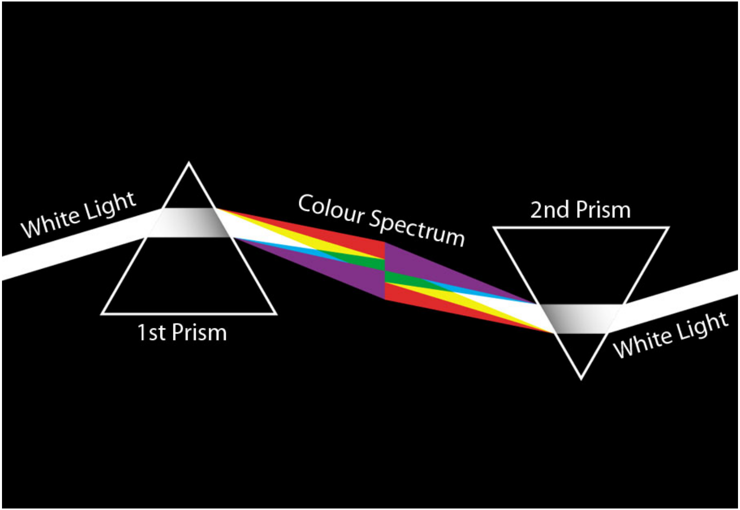


The diagram from Sir Isaac Newton's crucial experiment, 1666-72. A ray of light is divided into its constituent colors by the first prism (left), and the resulting bundle of colred rays is reconstituted into white light by the second.

ENLARGE

In the late 1660s, Newton starts experimenting with his '*celebrated phenomenon of colors*.' At the time, people thought that color was a mixture of light and darkness, and that prisms *colored* light. Hooke was a proponent of this theory of color, and had a scale that went from brilliant red, which was pure white light with the least amount of darkness added, to dull blue, the last step before black, which was the complete extinction of light by darkness. Newton realizes this theory was false.

Newton & Spectrum, II



At Last: Meaning of the Term “White Noise”

- **Newton’s Prism**:
  White light is made of **colored light**, in **equal** intensities.
- **Spectrum analysis**:
  White noise is made of **sinusoids** of different frequencies, in **equal intensities**.
- **Optical analogy**:

  "Colored Light" $\leftrightarrow$ “Sinosuid”

- **Acoustic analogy**:

  "Pure Tones" (e.g. middle A) $\leftrightarrow$ “Sinosuid”(e.g. 440Hz)

  Acoustic “White Noise” is a superposition of **all possible pure tones**, in **equal**, random amounts.

Noise Color

The optimal analogy suggests the following terminology:

- **Pink** noise:
  $f_x(\omega)$ is **large** near $\omega = 0$, i.e., x_t is 'built from' **lower** frequencies.
- **Blue** noise:
  $f_x(\omega)$ is **large** near $\omega = 1/2$, i.e., (x_t) is 'built from' **higher** frequencies.
- `https://en.wikipedia.org/wiki/Colors_of_noise`

Properties of Spectral Density, III

- Property 4.4 **Spectral density of ARMA**. (x_t) is ARMA(p, q):

$$f_x(\omega) = \sigma_w^2 \frac{\left|\theta(e^{-2\pi i\omega})\right|^2}{\left|\phi(e^{-2\pi i\omega})\right|^2},$$

  where:

  - $\phi(z) = 1 - \sum_{k=1}^{p} \phi_k z^k$ is **AR** polynomial.
  - $\theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k$ is **MA** polynomial.

- Example: (x_t) is **MA**(1),

$$f_x(\omega) = \sigma_w^2 \left|1 + \theta e^{-2\pi i\omega}\right|^2 = \sigma_w^2 \left(1 + 2\theta\cos(2\pi\omega) + \theta^2\right)$$

- Example: (x_t) is **AR**(1),

$$f_x(\omega) = \frac{\sigma_w^2}{1 + 2\phi\cos(2\pi\omega) + \phi^2}$$

Possible "Colors" of MA(1)

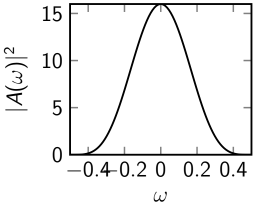
- Example: (x_t) is **MA**(1),

$$f_x(\omega) = \sigma_w^2 \left|1 + \theta e^{-2\pi i \omega}\right|^2 = \sigma^2 \left(1 + 2\theta \cos(2\pi\omega) + \theta^2\right)$$

- Pick $\theta = 1$:

$$f_x(\omega) = 2 + 2\cos(2\pi\omega)$$

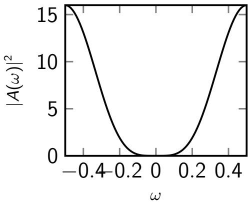
('Pink' Noise)



- Pick $\theta = -1$:

$$f_x(\omega) = 2 - 2\cos(2\pi\omega)$$

('Blue' Noise)



Possible "Colors" of AR(1)

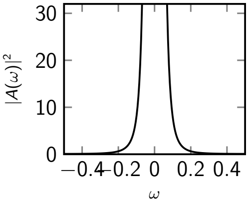
- Example: (x_t) is **AR**(1),

$$f_x(\omega) = \frac{\sigma^2}{1 + 2\theta\cos(2\pi\omega) + \theta^2}$$

- Pick $\phi = 1 - \epsilon$, $\epsilon > 0$ tiny (high positive correlation):

$$f_x(\omega) = \frac{\sigma_w^2}{1+(1-\epsilon)^2-2(1-\epsilon)\cos(2\pi\omega)}$$

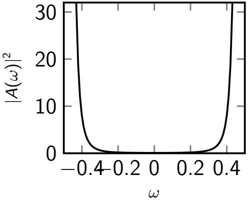
('Red' Noise)



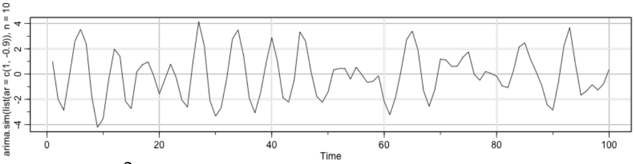
- Pick $\phi = -(1 - \epsilon)$, $\epsilon > 0$ tiny (high negative correlation):

$$f_x(\omega) = \frac{\sigma_w^2}{1+(1-\epsilon)^2+2(1-\epsilon)\cos(2\pi\omega)}$$

('Violet' Noise)

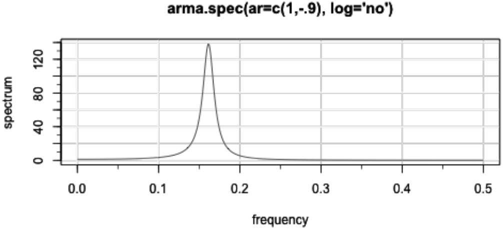


$$x_t - x_{t-1} + .9x_{t-2} = w_t, \qquad \sigma_w^2 = 1.$$



$\phi(z) = 1 - z + .9z^2$. $\theta(z) = 1$. From Property 4.4

$$f_x(\omega) = \left|\phi\left(e^{-2\pi i\omega}\right)\right|^{-2} = \left|1 - e^{-2\pi i\omega} + .9e^{-4\pi i\omega}\right|^{-2}$$
$$= (2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega))^{-1}$$



Linear Filters and Spectral Density, I

- Definition: **Linear filtering** of (x_t) to produce (y_t)

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

"(y_t) is the **convolution** of x_t and (a_t)".

- Definition: $(a_t)_{t \in \mathbb{Z}}$ is the filter's **impulse response function**.
- Definition: The filter's **frequency response function** is

$$A(\omega) \equiv \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i \omega j}.$$

- Property 4.3: If (x_t) has spectrum $f_x(\omega)$, then

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega).$$

Linear Filters and Spectral Density, II

- Example: Differencing

$$y_t = \nabla x_t$$

  - Frequency response

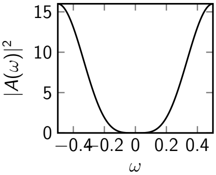
$$A(\omega) = 1 - e^{-2\pi i \omega}.$$

  - Relation between spectra

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega) = \left|1 - e^{-2\pi i \omega}\right|^2 f_x(\omega) = 2\left(1 - \cos\left(2\pi\omega\right)\right)^2 f_x(\omega).$$

  - Example: x_t is white noise with intensity σ^2:

$$f_y(\omega) = |A(\omega)|^2 \sigma^2 = 2\left(1 - \cos\left(2\pi\omega\right)\right)^2 \sigma^2$$



"Differencing white noise creates a bluish noise."

Linear Filters and Spectral Density, III

- Example: Symmetric Moving Average:

$$(a_t) = \left(\ldots, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, \ldots\right)$$

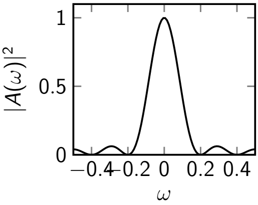
$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j} = \frac{1}{5}\left(x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2}\right).$$

  - Frequency response:

$$A(\omega) = \frac{1}{5}\left[1 + 2\cos(2\pi\omega) + 2\cos(4\pi\omega)\right]$$

x_t is white noise of intensity σ^2:

$$f_y(\omega) = |A(\omega)|^2 \sigma^2$$



  - "Moving average of white noise creates a pinkish noise."

Cross-Spectra

Cross-Covariance

- **Recall**: The **cross-covariance** of two stationary processes (x_t) and (y_t) is

$$\gamma_{xy}(h) = \mathrm{Cov}(x_{t+h}, y_t).$$

- Example: Delay + noise:

$$y_t = a \cdot x_{t-d} + w_t, \qquad (x_t) \text{ is stationary}$$

$$\begin{aligned}\gamma_{xy}(h) &= \mathrm{Cov}(x_{t+h}, a \cdot x_{t-d} + w_t) \\ &= a \cdot \mathrm{Cov}(x_{t+h}, x_{t-d}) = a\gamma_x(h+d).\end{aligned}$$

Cross-Spectral Density

- Definition: For two stationary processes (x_t) and (y_t), suppose that

$$\sum_{h=-\infty}^{\infty} |\gamma_{xy}(h)| < \infty.$$

Then the Fourier series

$$f_{xy}(\omega) = \sum_{h=-\infty}^{\infty} \gamma_{xy}(h) e^{-2\pi i \omega h},$$

defines a continuous **complex-valued** function on $(-1/2, 1/2)$, denoted the **cross-spectral density**.

- $\gamma_{xy}(h)$ can be recovered from

$$\gamma_{xy}(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \omega h} f_{xy}(\omega) d\omega, \qquad h = 0, \pm 1, \pm 2, \ldots .$$

(Fourier coefficients of $f_{xy}(\omega)$)

Properties of Cross-Spectral Density

- Warning: $f_{xy}(\omega)$ is, in general, complex-valued .
- **Real/Imaginary Decomposition**:

$$f_{xy}(\omega) = \overbrace{c_{xy}(\omega)}^{cospectrum} - i \overbrace{q_{xy}(\omega)}^{quadspectrum}, \qquad \omega \in (-1/2, 1/2).$$

- Hermitian Symmetry:

$$f_{xy}(\omega) = \overline{f_{yx}(\omega)},$$

$$c_{xy}(\omega) = c_{yx}(\omega), \qquad q_{xy}(\omega) = -q_{yx}(\omega).$$

(why?)

Coherence

- Definition: **Squared Coherence** function

$$\rho^2_{xy}(\omega) = \frac{|f_{yx}(\omega)|^2}{f_x(\omega)f_y(\omega)}$$

  (note similarity to correlation).

- Range:

$$0 \le \rho^2_{xy}(\omega) \le 1.$$

- Interpretation:
  - $\rho = 1$ implies perfect linear correlation at frequency ω.
  - $\rho = 0$ implies uncorrelatedness at frequency ω.

Cross-Spectral Density, Example

Delay + noise:

$$y_t = x_{t-d} + w_t, \qquad (w_t) \text{ is white noise independent of } (x_t).$$

- Cross-spectrum:

$$\begin{aligned} f_{xy}(\omega) &= \sum_{h=-\infty}^{\infty} \gamma_{xy}(h) e^{-2\pi i h \omega} = \sum_{h=-\infty}^{\infty} \gamma_x(h+d) e^{-2\pi i h \omega} \\ &= \sum_{u=-\infty}^{\infty} \gamma_x(u) e^{-2\pi i (u-d)\omega} = e^{2\pi i d \omega} \sum_{u=-\infty}^{\infty} \gamma_x(u) e^{-2\pi i u \omega} \\ &= e^{2\pi i d \omega} f_x(\omega). \end{aligned}$$

- **Amplitude** of cross-spectrum:

$$|f_{xy}(\omega)| = |f_x(\omega)| = f_x(\omega).$$

Cross-Spectral Density, Example (cont'd)

Delay + noise:

$$y_t = x_{t-d} + w_t, \qquad (w_t) \text{ white noise independent of } (x_t).$$

- Spectral density of (y_t):

$$f_y(\omega) = f_x(\omega) + f_w(\omega) = f_x(\omega) + \sigma_w^2$$

- Squared Coherence:

$$\rho_{xy}^2(\omega) = \frac{|f_{xy}(\omega)|^2}{f_x(\omega) f_y(\omega)} = \frac{|f_x(\omega)|^2}{f_x(\omega) f_y(\omega)} = \frac{f_x(\omega)}{f_x(\omega) + \sigma_w^2}$$

- **Signal-to-Noise Ratio** (SNR):

$$\text{SNR}(\omega) \equiv \frac{f_x(\omega)}{f_w(\omega)} = \frac{f_x(\omega)}{\sigma_w^2} \geq 0.$$

- Squared Coherence in terms of SNR:

$$\rho_{xy}^2(\omega) = \frac{\text{SNR}(\omega)}{1 + \text{SNR}(\omega)} \in [0, 1], \qquad \omega \in (-1/2, 1/2).$$

Linear Filters and Cross Spectra, I

- **Recall**: Linear filtering:

$$y_t = \sum_{h=-\infty}^{\infty} a_h x_{t-h},$$

  where $(a_t)_{t\in\mathbb{Z}}$ is absolutely summable ($(a_t)_{t\in\mathbb{Z}}$ is the **impulse response** of the filter).

- The spectral density of the filter's **output** (Property 4.3):

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega),$$

  where

$$A(\omega) = \sum_{h=-\infty}^{\infty} a_h e^{-2\pi i\omega h}, \qquad \omega \in (-1/2, 1/2).$$

- Q: What is the input-output **cross-spectrum** $f_{xy}(\omega)$?

- A: $f_{yx}(\omega) = A(\omega) f_x(\omega)$.

Linear Filters and Cross Spectra, Example

- Example: Pure delay

$$y_t = a \cdot x_{t-d}$$

- Frequency response

$$A(\omega) = a \cdot e^{-2\pi i d\omega}.$$

- Cross-spectrum:

$$f_{yx}(\omega) = a \cdot e^{-2\pi i d\omega} f_x(\omega).$$

- Output spectrum

$$f_y(\omega) = a^2 \cdot f_x(\omega)$$

- Squared coherence

$$\rho^2_{yx}(\omega) = \frac{\left|a \cdot e^{-2\pi i d\omega}\right|^2}{a^2 f_x(\omega) \cdot f_x(\omega)} = 1.$$

"Time-delay does not affect correlation at frequency ω, for all $\omega \in (-1/2, 1/2)$."

Linear Filters and Cross Spectra, Example 4.19

- Example: Three-point moving average

$$y_t = \frac{1}{3}\left(x_{t-1} + x_t + x_{t+1}\right)$$

- Frequency response

$$A(\omega) = \frac{1}{3}\left(1 + 2\cos(2\pi\omega)\right).$$

- Cross-spectrum:

$$f_{yx}(\omega) = \frac{1}{3}\left(1 + 2\cos(2\pi\omega)\right) f_x(\omega).$$

(purely real!)

- Output spectrum:

$$f_y(\omega) = \frac{1}{9}\left(1 + 2\cos(2\pi\omega)\right)^2 f_x(\omega).$$

- Squared coherence:

$$\rho^2_{xy} = \frac{\left|\frac{1}{3}\left(1 + 2\cos(2\pi\omega)\right) f_x(\omega)\right|^2}{f_x(\omega) \cdot \frac{1}{9}\left(1 + 2\cos(2\pi\omega)\right)^2 f_x(\omega)} = 1.$$

Recap

- Periodogram indicates the **component of data variance explainable by sinusoids** at frequency j.
- The **spectral density** $f(\omega)$ has a **Fourier series** representation with coefficients given by the **covariance function** $\gamma(h)$.
- The **spectral density** gives **typical size** of random variable **periodogram**.

- The **cross-spectral density** $f_{xy}(\omega)$ has a **Fourier series** representation with coefficients given by the **cross covariance function** $\gamma_{xy}(h)$.

- The **spectral density** and **cross-spectral density** play nicely with linear filtering.

Next 1-2 Lectures:

- Spectral **estimation**.
- Frequency domain **regression** & principal components analysis.